

Unitarizable weight modules over generalized Weyl algebras

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March 5, 2008

Abstract

We define a notion of unitarizability for weight modules over a generalized Weyl algebra (of rank one, with commutative coefficient ring R), which is assumed to carry an involution of the form $X^* = Y$, $R^* \subseteq R$. We prove that a weight module V is unitarizable iff it is isomorphic to its finitistic dual $V^\#$. Using the classification of weight modules by Drozd, Guzner and Ovsienko, we obtain necessary and sufficient conditions for an indecomposable weight module to be isomorphic to its finitistic dual, and thus to be unitarizable. Some examples are given, including $U_q(\mathfrak{sl}_2)$ for q a root of unity.

1 Introduction

For a $*$ -algebra A over \mathbb{C} and an A -module V , a basic question is whether V is unitarizable. That is, can V be equipped with an inner product which is A -admissible, i.e. $(av, w) = (v, a^*w)$ for $a \in A, v, w \in V$? This is so in many well-behaved examples, like simple finite-dimensional modules over a finite-dimensional group-algebra, but unfortunately false in general. However, the modules for which this is false might still be unitarizable in the weaker sense of having an admissible inner product which is non-degenerate but not necessarily positive definite.

A new feature for this broadened notion of unitarizability is that there may exist unitarizable indecomposable modules which are not simple.

Such indefinite inner product spaces have been thoroughly studied in the analytical setting of operator algebras, see [KS]. There are also many applications to areas in physics, for example quantum field theory. See [MS] and references therein.

On the algebraic side, existence and uniqueness questions of such indefinite inner products was considered in [MT1] in the general situation of A being a $*$ -algebra over an algebraically closed field and M being a finite-dimensional A -module, or a weight A -module with finite-dimensional weight spaces. Among other things, it was shown that an A -module M has a non-degenerate admissible form iff M is isomorphic to its finitistic dual $M^\#$. In [MT2] the authors

described all simple weight (with respect to a Cartan subalgebra) modules with finite-dimensional weight spaces over a complex finite-dimensional semisimple Lie algebra which are unitarizable with a non-degenerate symmetric form.

In this paper we consider generalized Weyl algebras (GWAs). These are certain noncommutative rings, first introduced in [Bav], and studied since in many different papers (see [BB], [BO], [BL] and references therein). The class contains a wide range of examples such as ambiskew polynomial rings [Jor], which includes Noetherian generalized down-up algebras [CS]; $U(\mathfrak{sl}_2)$ and its various deformations and generalizations (see for example [BO]) as well as the first Weyl algebra and quantum Weyl algebra.

We will consider GWAs of rank one, $A = R(\sigma, t)$, and assume that R is a commutative ring. For such GWAs, all indecomposable weight modules with finite-dimensional weight spaces were classified in [DGO], up to indecomposable elements in a skew polynomial ring over a field. There are five families of modules, some of them depending on many parameters. It is interesting, therefore, to ask if some of these modules possess extra structure.

The purpose of this paper is two-fold:

- 1) To define an appropriate notion of unitarizability for weight modules over a generalized Weyl algebra equipped with an involution satisfying $X^* = Y$, $Y^* = X$, $R^* \subseteq R$. See Definition 3.1.
- 2) To find conditions on the parameters of the indecomposable weight modules V over a generalized Weyl algebra, which are necessary and sufficient for the modules to be unitarizable with a non-degenerate admissible form. The main results here are Theorems 5.2, 5.3, 5.6, 5.8, and 5.13 which completely answers this question in the case of real orbit ω , i.e. $\mathfrak{m}^* = \mathfrak{m} \forall \mathfrak{m} \in \omega$.

After recalling some basic definitions in Section 2, we give in Section 3 the definition of admissible form and of the finitistic dual V^\sharp . We prove analogs of some results from [MT1] such as Proposition 3.18 on the correspondence between forms and morphisms.

In Section 4 we recall the classification theorem from [DGO]. We have collected all notation necessary in Section 4.1.

In Section 5 we consider in turn each type of indecomposable weight module and give necessary and sufficient conditions for the existence of a non-degenerate admissible form.

We end by considering some examples in Section 6. In particular we obtain in Section 6.3 conditions for indecomposable non-simple modules over $U_q(\mathfrak{sl}_2)$ (q a root of unity), to have non-degenerate admissible forms.

2 Setup

Let

- R be a commutative ring with 1,

- $*$: $R \rightarrow R$ an automorphism of order 1 or 2,
- $\sigma : R \rightarrow R$ an automorphism commuting with $*$, and
- $t \in R$ be selfadjoint, i.e. $t^* = t$.

Let $A = R(\sigma, t)$ be the associated *generalized Weyl algebra* (GWA) [Bav]. Thus A is the ring generated by the set $R \cup \{X, Y\}$, where X, Y are two new symbols, with the relations that R is a subring of A and

$$YX = t, \quad XY = \sigma(t), \quad Xr = \sigma(r)X, \quad Yr = \sigma^{-1}(r)Y \quad \forall r \in R. \quad (2.1)$$

By (2.1), $*$ extends to an involution on A (i.e. $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$, $a^{**} = a$, $\forall a, b \in A$) by requiring

$$X^* = Y, \quad Y^* = X.$$

Relations (2.1) also imply that A is a \mathbb{Z} -graded ring $A = \bigoplus_{n \in \mathbb{Z}} A_n$ with gradation given by $\deg X = 1, \deg Y = -1, \deg r = 0 \forall r \in R$. Let Ω be the set of orbits for the action of σ on the set $\text{Max}(R)$ of maximal ideals of R . For $\omega \in \Omega$ we let R_ω denote the direct sum of all the R -modules R/\mathfrak{m} for $\mathfrak{m} \in \omega$:

$$R_\omega = \bigoplus_{\mathfrak{m} \in \omega} R/\mathfrak{m}. \quad (2.2)$$

The R -module R_ω will be used as a substitute for a ground field, when defining admissible forms in Section 3.2. The automorphism σ induces isomorphisms $R/\mathfrak{m} \rightarrow R/\sigma(\mathfrak{m})$, $\mathfrak{m} \in \text{Max}(R)$, which we also denote by σ . Extending additively, we get a map $\sigma : R_\omega \rightarrow R_\omega$. The automorphism $*$ of R induces a map $R/\mathfrak{m} \rightarrow R/\mathfrak{m}^*$, and hence a map $R_\omega \rightarrow R_{\omega^*}$ which will be denoted by conjugation.

Remark 2.1. Let $A = R(\sigma, t)$ be a GWA and $*$ an anti-involution on A satisfying $R^* \subseteq R$ and $X^* = \varepsilon Y$, where $\varepsilon \in R$ is invertible. Then, after a change of generators, we can assume $\varepsilon = 1$ and thus that $t^* = t$. Indeed, set $X_1 = X$, $Y_1 = \varepsilon Y$ and $t_1 = Y_1 X_1 = \varepsilon t$. Then $X_1 Y_1 = X \varepsilon Y = \sigma(\varepsilon) \sigma(t) = \sigma(t_1)$. Clearly $X_1 r = \sigma(r) X_1$ and $Y_1 r = \sigma^{-1}(r) Y_1$, $\forall r \in R$. Moreover $X_1^* = Y_1$ so that $t_1^* = t_1$.

Definition 2.2. A module V over a ring, which contains R as a subring, will be called a *weight module* if $V = \bigoplus_{\mathfrak{m} \in \text{Max}(R)} V_{\mathfrak{m}}$, where $V_{\mathfrak{m}} = \{v \in V : \mathfrak{m}v = 0\}$. The R -submodules $V_{\mathfrak{m}}$ of V are called *weight spaces* and elements of $V_{\mathfrak{m}}$ are *weight vectors of weight \mathfrak{m}* . The *support* of V , denoted $\text{Supp}(V)$, is defined as the set $\{\mathfrak{m} \in \text{Max}(R) : V_{\mathfrak{m}} \neq 0\}$.

3 Admissible forms and the finitistic dual

3.1 Motivation of definition

In section 3.2 we will define an admissible form on a weight A -module V to be a certain biadditive form on V with values in the R -module R_ω . To motivate

this definition, let us first consider another, at first sight more natural, attempt at a definition.

As we will see, a problem appears when ω is finite. Suppose therefore that $\omega \in \Omega$ is a finite orbit. Let $p = |\omega|$. Let $\omega \in \Omega$ and let V be a weight module over A with $\text{Supp}(V) \subseteq \omega$. If we choose and fix an element $\mathfrak{m} \in \omega$, we can define a R/\mathfrak{m} -vector space structure on V by $(r + \mathfrak{m})v = \sigma^k(r)v$ if $v \in V_{\sigma^k(\mathfrak{m})}$ and $0 \leq k < p$. Then, for $v \in V_{\sigma^k(\mathfrak{m})}$ and $\lambda = r + \mathfrak{m} \in R/\mathfrak{m}$,

$$X^p \lambda v = X^p \sigma^k(r)v = \sigma^{p+k}(r)X^p v = \sigma^p(\lambda)X^p v.$$

It would perhaps seem natural to define V to be unitarizable if there is a nonzero *admissible* R/\mathfrak{m} -form on V , i.e. a map $G : V \times V \rightarrow R/\mathfrak{m}$ satisfying

$$G \text{ is additive in each argument,} \quad (3.1a)$$

$$G(\lambda v, w) = \lambda G(v, w) \quad \text{for all } v, w \in V, \lambda \in R/\mathfrak{m}, \quad (3.1b)$$

$$G(av, w) = G(v, a^* w) \quad \text{for all } v, w \in V, a \in A. \quad (3.1c)$$

However, then, for $v, w \in V$ and $\lambda \in R/\mathfrak{m}$,

$$G(X^p \lambda v, w) = G(\lambda v, Y^p w) = \lambda G(v, Y^p w) = \lambda G(X^p v, w),$$

while on the other hand,

$$G(X^p \lambda v, w) = G(\sigma^p(\lambda)X^p v, w) = \sigma^p(\lambda)G(X^p v, w).$$

Thus, any weight module V with $\text{Supp}(V) \subseteq \omega$ on which $X^p \neq 0$ (or $Y^p \neq 0$ for analogous reasons) would automatically be excluded from the possibility of being unitarizable (at least with a non-degenerate form), unless $\sigma^p : R/\mathfrak{m} \rightarrow R/\mathfrak{m}$ is the identity map for some (hence all) $\mathfrak{m} \in \omega$.

Although $\sigma^p : R/\mathfrak{m} \rightarrow R/\mathfrak{m}$ is the identity in many important examples (for example, if R is a finitely generated algebra over an algebraically closed field k and σ is a k -algebra automorphism, then $\sigma^p : R/\mathfrak{n} \rightarrow R/\mathfrak{n}$ is the identity for any $\mathfrak{n} \in \text{Max}(R)$ with $\sigma^p(\mathfrak{n}) = \mathfrak{n}$), we feel that this notion of admissible form is too restrictive.

To remedy this situation we introduce in Section 3.2 a modified definition of unitarizability which has three advantages. First, no unnecessary restrictions applies as to which modules can be unitarizable when $\sigma^p : R/\mathfrak{m} \rightarrow R/\mathfrak{m}$ is nontrivial. Secondly, the definition does not depend on any unnatural choice of maximal ideal in the orbit. And thirdly, in the special case when $\sigma^p : R/\mathfrak{m} \rightarrow R/\mathfrak{m}$ really is the identity map (and also when the orbit ω is infinite), the definition is equivalent to the one above in the sense that one form can be obtained from the other in a bijective manner, as described in Proposition 3.4.

3.2 Admissible forms and unitarizability

Let $\omega \in \Omega$ and V be a weight module over A with $\text{Supp}(V) \subseteq \omega$.

Definition 3.1. An *admissible form* F on V is a map

$$F : V \times V \rightarrow R_\omega$$

such that

$$F \text{ is additive in each argument,} \quad (3.2a)$$

$$F(rv, w) = rF(v, w) \quad \text{for all } v, w \in V, r \in R, \quad (3.2b)$$

$$F(av, w) = \sigma^{\deg a}(F(v, a^*w)) \quad \text{for all } v, w \in V, a \in \cup_{n \in \mathbb{Z}} A_n. \quad (3.2c)$$

An admissible form F is called *non-degenerate* if for any nonzero $v \in V$ there exist $w_1, w_2 \in V$ such that $F(w_1, v) \neq 0 \neq F(v, w_2)$.

Definition 3.2. A weight module V over A , whose support is contained in an orbit, is *unitarizable* if there exists a nonzero admissible form on V .

Note that, since $\deg a^* = -\deg a$ for homogenous $a \in A$, relation (3.2c) is equivalent to $F(v, aw) = \sigma^{\deg a}(F(a^*v, w))$.

3.3 Relation to admissible R/\mathfrak{m} -forms

In view of the discussion in Section 3.1 we make the following definition.

Definition 3.3. We call $\omega \in \Omega$ *torsion trivial* if whenever $\mathfrak{m} \in \omega$, $n \in \mathbb{Z}$ and $\sigma^n(\mathfrak{m}) = \mathfrak{m}$ then the induced map $\sigma^n : R/\mathfrak{m} \rightarrow R/\mathfrak{m}$ is the identity.

Assume that $\omega \in \Omega$ is torsion trivial. For $\mathfrak{m}_1, \mathfrak{m}_2 \in \omega$, say $\mathfrak{m}_2 = \sigma^n(\mathfrak{m}_1)$, define $\sigma_{\mathfrak{m}_1, \mathfrak{m}_2} = \sigma^n : R/\mathfrak{m}_1 \rightarrow R/\mathfrak{m}_2$. Then $\sigma_{\mathfrak{m}_1, \mathfrak{m}_2}$ is independent of the choice (if any) of n , since ω is torsion trivial. Fix $\mathfrak{m} \in \omega$. Let V be a weight A -module with $\text{Supp}(V) \subseteq \omega$. Give V the structure of an R/\mathfrak{m} -vector space by $(r + \mathfrak{m})v = \sigma_{\mathfrak{m}, \sigma^k(\mathfrak{m})}(r + \mathfrak{m})v = \sigma^k(r)v$ for $v \in V_{\sigma^k(\mathfrak{m})}$ and $r + \mathfrak{m} \in R/\mathfrak{m}$.

Proposition 3.4. When ω is torsion trivial, there is a bijective correspondence between admissible forms F and admissible R/\mathfrak{m} -forms G on V .

Proof. Given F , define G by $G = \pi \circ F$, where $\pi : R_\omega \rightarrow R/\mathfrak{m}$ is given by

$$\pi((\lambda_n)_{n \in \omega}) = \sum_{\mathfrak{n} \in \omega} \sigma_{\mathfrak{n}, \mathfrak{m}}(\lambda_n).$$

Since F is biadditive, so is G . To verify (3.1b), let $\mathfrak{n} = \sigma^k(\mathfrak{m}) \in \omega$ be arbitrary, $v \in V_{\sigma^k(\mathfrak{m})}$, $w \in V$ and $\lambda = r + \mathfrak{m} \in R/\mathfrak{m}$. Then, using that $F(V_{\mathfrak{n}}, V) \subseteq R/\mathfrak{n}$, which follows from (3.2b), we have

$$\begin{aligned} G(\lambda v, w) &= \pi(F(\sigma^k(r)v, w)) = \sigma^{-k}(\sigma^k(r)F(v, w)) = r\sigma^{-k}(F(v, w)) = \\ &= \lambda G(v, w). \end{aligned}$$

To show (3.1c), let $\mathfrak{n} \in \omega$, $v \in V_{\mathfrak{n}}$, $a \in A_k$. Then $av \in V_{\sigma^k(\mathfrak{n})}$ so

$$\begin{aligned} G(av, w) &= \sigma_{\sigma^k(\mathfrak{n}), \mathfrak{m}}(F(av, w)) = \sigma_{\sigma^k(\mathfrak{n}), \mathfrak{m}}\sigma^k(F(v, a^*w)) = \sigma_{\mathfrak{n}, \mathfrak{m}}(F(v, a^*w)) = \\ &= G(v, a^*w). \end{aligned}$$

This proves that G is an admissible R/\mathfrak{m} -form on V .

Conversely, given G , define F by

$$F(v, w) = \sigma_{\mathfrak{m}, \mathfrak{n}}(G(v, w)) \quad \text{for } v \in V_{\mathfrak{n}}, w \in V.$$

Then F is biadditive. To prove (3.2b), let $\mathfrak{n} = \sigma^k(\mathfrak{m}) \in \omega, v \in V_{\mathfrak{n}}, w \in V$ and $r \in R$. Put $\lambda = r + \mathfrak{m}$. We have

$$\begin{aligned} F(\sigma^k(r)v, w) &= \sigma^k(G(\sigma^k(r)v, w)) = \sigma^k(G(\lambda v, w)) = \sigma^k(\lambda G(v, w)) = \\ &= \sigma^k(r)\sigma^k(G(v, w)) = \sigma^k(r)F(v, w). \end{aligned}$$

Since r was arbitrary, (3.2b) is proved. It remains to show that F satisfies (3.2c). Let $v \in V_{\mathfrak{n}}, a \in A_k$. Then

$$F(av, w) = \sigma_{\mathfrak{m}, \sigma^k(\mathfrak{n})}(G(av, w)) = \sigma^k \circ \sigma_{\mathfrak{m}, \mathfrak{n}}(G(v, a^*w)) = \sigma^k(F(v, a^*w)).$$

Thus F is an admissible form on V . \square

3.4 Symmetric and real orbits

Definition 3.5. An orbit $\omega \in \Omega$ is called *symmetric* if $\mathfrak{m}^* \in \omega$ for any $\mathfrak{m} \in \omega$, and *real* if $\mathfrak{m}^* = \mathfrak{m}$ for any $\mathfrak{m} \in \omega$.

Proposition 3.6. *If ω is symmetric but not real, then $|\omega|$ is finite, even, $|\omega| \geq 4$, and $\mathfrak{m}^* = \sigma^{|\omega|/2}(\mathfrak{m})$ for any $\mathfrak{m} \in \omega$.*

Proof. Since ω is symmetric but not real, there is some $\mathfrak{n} \in \omega$ such that $\mathfrak{n}^* = \sigma^N(\mathfrak{n})$ for some $N \neq 0$. Then

$$\mathfrak{n} = \mathfrak{n}^{**} = \sigma^N(\mathfrak{n})^* = \sigma^N(\mathfrak{n}^*) = \sigma^{2N}(\mathfrak{n}).$$

Hence $|\omega| = p < \infty$ and $2N$ is a multiple of p . Without loss of generality we can assume $0 < N < p$. Then $2N = p$ is the only possibility. Thus $|\omega| \geq 4$ and $\mathfrak{n}^* = \sigma^{|\omega|/2}(\mathfrak{n})$. Since any $\mathfrak{m} \in \omega$ has the form $\sigma^k(\mathfrak{n})$, and σ and $*$ commute, it follows that $\mathfrak{m}^* = \sigma^{|\omega|/2}(\mathfrak{m})$ for any $\mathfrak{m} \in \omega$. \square

3.5 Orthogonality of weight spaces

Proposition 3.7. *Let $\omega \in \Omega$ and let V be a weight A -module with $\text{Supp}(V) \subseteq \omega$. If F is an admissible form on V , then $F(V_{\mathfrak{m}}, V_{\mathfrak{n}}) = 0$ for any $\mathfrak{m}, \mathfrak{n} \in \omega$ with $\mathfrak{m} \neq \mathfrak{n}^*$.*

Proof. By (3.2b) and (3.2c),

$$(\mathfrak{m} + \mathfrak{n}^*)F(V_{\mathfrak{m}}, V_{\mathfrak{n}}) = F(\mathfrak{m}V_{\mathfrak{m}}, V_{\mathfrak{n}}) + F(V_{\mathfrak{m}}, \mathfrak{n}V_{\mathfrak{n}}) = 0.$$

If $\mathfrak{m} \neq \mathfrak{n}^*$ then $\mathfrak{m} + \mathfrak{n}^* = R \ni 1$ so $F(V_{\mathfrak{m}}, V_{\mathfrak{n}}) = 0$. \square

Corollary 3.8. *Let $\omega \in \Omega$ be an orbit. If there exists a unitarizable weight A -module V with $\text{Supp}(V) \subseteq \omega$, then ω is symmetric.*

Proof. If V is unitarizable, it has a nonzero admissible form F . Since F is nonzero and V is a weight module, $F(V_{\mathbf{m}}, V_{\mathbf{n}}) \neq 0$ for some $\mathbf{m}, \mathbf{n} \in \text{Supp}(V) \subseteq \omega$. By Proposition 3.7, $\mathbf{m}^* = \mathbf{n} \in \omega$. If $\mathbf{m}_1 \in \omega$ is arbitrary, then $\mathbf{m}_1 = \sigma^n(\mathbf{m})$ for some n and $\mathbf{m}_1^* = \sigma^n(\mathbf{m})^* = \sigma^n(\mathbf{m}^*) = \sigma^n(\mathbf{n}) \in \omega$. This proves that ω is symmetric. \square

Corollary 3.9. *If $\omega \in \Omega$ is real and V is a weight A -module with $\text{Supp}(V) \subseteq \omega$, then the weight spaces of V are pairwise orthogonal with respect to any admissible form.*

Proof. This is immediate from Proposition 3.7. \square

3.6 The finitistic dual V^\sharp

Let $\omega \in \Omega$ and V be a weight module over A with $\text{Supp}(V) \subseteq \omega$. Suppose F is an admissible form on V . Let $u \in V$. Define $\tilde{F}_u : V \rightarrow R_\omega$ by $\tilde{F}_u(v) = F(u, v)$.

Proposition 3.10. *The map \tilde{F}_u has the following properties:*

$$\tilde{F}_u(v_1 + v_2) = \tilde{F}_u(v_1) + \tilde{F}_u(v_2) \quad \forall v_1, v_2 \in V, \quad (3.3a)$$

$$\tilde{F}_u(rv) = r^* \tilde{F}_u(v) \quad \forall r \in R, v \in V, \quad (3.3b)$$

$$\tilde{F}_u(V_{\mathbf{m}}) = 0 \quad \text{for all but finitely many } \mathbf{m} \in \omega. \quad (3.3c)$$

Proof. (3.3a), (3.3b) follow from (3.2a)-(3.2c). For (3.3c), write $u = \sum_{i=1}^n u_i$, where $u_i \in V_{\mathbf{m}_i}$. Then if $\mathbf{n} \in \omega \setminus \{\mathbf{m}_1^*, \dots, \mathbf{m}_n^*\}$ we get

$$\tilde{F}_u(V_{\mathbf{n}}) = F(u_1, V_{\mathbf{n}}) + \dots + F(u_n, V_{\mathbf{n}}) = 0$$

by Proposition 3.7. \square

Definition 3.11. Let $\omega \in \Omega$ and V be a weight A -module with $\text{Supp}(V) \subseteq \omega$. The *finitistic dual* V^\sharp of V is the set of all maps $\varphi : V \rightarrow R_\omega$ satisfying the properties of Proposition 3.10, i.e.

$$\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2) \quad \forall v_1, v_2 \in V, \quad (3.4a)$$

$$\varphi(rv) = r^* \varphi(v) \quad \forall r \in R, v \in V, \quad (3.4b)$$

$$\varphi(V_{\mathbf{m}}) = 0 \quad \text{for all but finitely many } \mathbf{m} \in \omega. \quad (3.4c)$$

Proposition 3.12. V^\sharp carries an A -module structure defined as follows. Let $\varphi \in V^\sharp$ and $r \in R$. Define $r\varphi, X\varphi, Y\varphi : V \rightarrow R_\omega$ by

$$(r\varphi)(v) = \varphi(r^*v) = r\varphi(v), \quad (3.5a)$$

$$(X\varphi)(v) = \sigma(\varphi(Yv)), \quad (3.5b)$$

$$(Y\varphi)(v) = \sigma^{-1}(\varphi(Xv)), \quad (3.5c)$$

for any $v \in V$.

Proof. First we must prove that $r\varphi, X\varphi, Y\varphi \in V^\sharp$. It is clear that $r\varphi$ satisfies (3.4a),(3.4b),(3.4c) since φ does. Also $X\varphi$ and $Y\varphi$ satisfies (3.4a),(3.4c). We show (3.4b) for $X\varphi$:

$$\begin{aligned} (X\varphi)(rv) &\stackrel{(3.5b)}{=} \sigma(\varphi(Yrv)) = \sigma(\varphi(\sigma^{-1}(r)Yv)) \stackrel{(3.4b)}{=} \sigma(\sigma^{-1}(r)^*)\sigma(\varphi(Yv)) = \\ &\stackrel{(3.5b)}{=} r^*(X\varphi)(v). \end{aligned}$$

Analogously, $Y\varphi$ satisfies (3.4b).

We must also show that the relations in A are preserved. For any $\varphi \in V^\sharp$ we have

$$(YX\varphi)(v) \stackrel{(3.5c)}{=} \sigma^{-1}((X\varphi)(Xv)) \stackrel{(3.5b)}{=} \varphi(YXv) = \varphi(tv) \stackrel{(3.5a)}{=} (t\varphi)(v) \quad \forall v \in V$$

so $YX\varphi = t\varphi$. Similarly, $XY\varphi = \sigma(t)\varphi$ for any $\varphi \in V^\sharp$. Also, for any $r \in R$ and $\varphi \in V^\sharp$,

$$\begin{aligned} (Xr\varphi)(v) &\stackrel{(3.5b)}{=} \sigma((r\varphi)(Yv)) \stackrel{(3.5a)}{=} \sigma(\varphi(r^*Yv)) = \sigma(\varphi(Y\sigma(r^*)v)) = \\ &\stackrel{(3.5b)}{=} (X\varphi)(\sigma(r)^*v) \stackrel{(3.5a)}{=} (\sigma(r)X\varphi)(v) \quad \forall v \in V. \end{aligned}$$

Analogously one proves that $Yr\varphi = \sigma^{-1}(r)Y\varphi$ for any $r \in R, \varphi \in V^\sharp$. Thus the relations of A are preserved, so (3.5a)-(3.5c) extends to an action of A on V^\sharp . \square

Proposition 3.13. V^\sharp is a weight A -module with

$$(V^\sharp)_{\mathfrak{m}} = \{\varphi \in V^\sharp : \varphi|_{V_{\mathfrak{n}}} = 0 \text{ for all } \mathfrak{n} \in \omega \text{ except possibly for } \mathfrak{n} = \mathfrak{m}^*\} \quad (3.6)$$

$$= \{\varphi \in V^\sharp : \varphi(V) \subseteq R/\mathfrak{m}\}. \quad (3.7)$$

Proof. Let $\varphi \in V^\sharp$. Then $\mathfrak{m}\varphi = 0 \Leftrightarrow \varphi(\mathfrak{m}^*v) = 0 \quad \forall v \in V \Leftrightarrow \varphi|_{V_{\mathfrak{n}}} = 0$ for all $\mathfrak{n} \in \omega$ except possibly for $\mathfrak{n} = \mathfrak{m}^*$, proving (3.6). The second equality holds since $\mathfrak{m}\varphi = 0 \Leftrightarrow \mathfrak{m}\varphi(V) = 0 \Leftrightarrow \varphi(V) \subseteq (R_{\omega})_{\mathfrak{m}} = R/\mathfrak{m}$. Since any φ is the sum of its corestrictions $\varphi_{\mathfrak{m}} = \pi_{\mathfrak{m}} \circ \varphi$, where $\pi_{\mathfrak{m}} : R_{\omega} \rightarrow R/\mathfrak{m}$, V^\sharp is a weight module. \square

Proposition 3.14. Let $\omega \in \Omega$ and let V be a weight A -module with $\text{Supp}(V) \subseteq \omega$. Then $\text{Supp}(V^\sharp) = \text{Supp}(V)^* = \{\mathfrak{m}^* : \mathfrak{m} \in \text{Supp}(V)\}$.

Proof. Assume $\mathfrak{m} \in \text{Supp}(V^\sharp)$ and let $0 \neq \varphi \in (V^\sharp)_{\mathfrak{m}}$. Then, by (3.6), $\varphi(v) \neq 0$ for some $v \in V_{\mathfrak{m}^*}$. This implies that $\mathfrak{m}^* \in \text{Supp}(V)$, i.e. $\mathfrak{m} \in \text{Supp}(V)^*$. Conversely, if $\mathfrak{m} \in \text{Supp}(V)^*$ and $0 \neq v \in V_{\mathfrak{m}^*}$ we can extend v to an R/\mathfrak{m}^* -basis of $V_{\mathfrak{m}^*}$ and define $\varphi \in V^\sharp$ by requiring that $\varphi(V_{\mathfrak{n}}) = 0$, $\mathfrak{n} \neq \mathfrak{m}^*$, $\varphi(v) = 1 + \mathfrak{m}$ and $\varphi(w) = 0$ for all other basis vectors w in $V_{\mathfrak{m}^*}$. Then, by (3.6), $\varphi \in (V^\sharp)_{\mathfrak{m}}$ so that $\mathfrak{m} \in \text{Supp}(V^\sharp)$. \square

Proposition 3.15. If $\dim_{R/\mathfrak{m}} V_{\mathfrak{m}} < \infty$ for all $\mathfrak{m} \in \text{Supp}(V)$ then $V^{\sharp\sharp}$ and V are isomorphic as A -modules.

Proof. Define $\Psi : V \rightarrow V^\sharp$ by $\Psi(v)(\varphi) = \varphi(v)$ for $v \in V, \varphi \in V^\sharp$. Then

$$\Psi(Xv)(\varphi) = \varphi(Xv) \stackrel{(3.5c)}{=} \sigma((Y\varphi)(v)) = \sigma(\Psi(v)(Y\varphi)) \stackrel{(3.5b)}{=} (X\Psi(v))(\varphi)$$

for any $v \in V, \varphi \in V^\sharp$. Similarly, $\Psi(Yv) = Y\Psi(v)$ and $\Psi(rv) = r\Psi(v)$ for any $r \in R$, proving that Ψ is an A -module homomorphism. Let $v \in V, v \neq 0$ and write v as a finite sum of weight vectors $v_m \neq 0$. Then there exists $\varphi \in (V^\sharp)_m^*$ such that $\varphi(v) \neq 0$, i.e. $\Psi(v)(\varphi) \neq 0$ so $\Psi(v) \neq 0$. Thus Ψ is injective. Also, by considering dual bases, $\dim V_m = \dim (V^\sharp)_m$. Since $\Psi(V_m) \subseteq (V^\sharp)_m$ we conclude that Ψ is an isomorphism. \square

Let $\omega \in \Omega$. If $\Psi : V \rightarrow W$ is a homomorphism of weight A -modules with support in ω , we define $\Psi^\sharp : W^\sharp \rightarrow V^\sharp$ by

$$(\Psi^\sharp(\varphi))(v) = \varphi(\Psi(v)) \quad \forall v \in V, \forall \varphi \in W^\sharp \quad (3.8)$$

Proposition 3.16. Ψ^\sharp is also an A -module homomorphism. Moreover, \sharp is a contravariant endofunctor on the category of weight A -modules with support in ω .

Proof. For any $v \in V, \varphi \in W^\sharp, r \in R$, we have

$$\begin{aligned} (\Psi^\sharp(r\varphi))(v) &= (r\varphi)(\Psi(v)) && \text{by definition of } \Psi^\sharp \\ &= \varphi(r^*\Psi(v)) && \text{by } A\text{-module structure on } W^\sharp \\ &= \varphi(\Psi(r^*v)) && \text{since } \Psi \text{ is an } A\text{-module morphism} \\ &= (\Psi^\sharp(\varphi))(r^*v) && \text{by definition of } \Psi^\sharp \\ &= (r\Psi^\sharp(\varphi))(v) && \text{by } A\text{-module structure on } V^\sharp \end{aligned}$$

In the same way one shows that Ψ^\sharp commutes with the actions of X and Y . That \sharp is a functor is easy to check. \square

3.7 The bijection between forms and morphisms

Let $\omega \in \Omega$ and V be a weight A -module with $\text{Supp}(V) \subseteq \omega$. Assume F is an admissible form on V . For $u \in V$, recall that $\tilde{F}_u \in V^\sharp$ by Proposition 3.10.

Proposition 3.17. The map $\tilde{F} : V \rightarrow V^\sharp$ defined by $u \mapsto \tilde{F}_u$ is an A -module homomorphism.

Proof. For any $r \in R, u, v \in V$ we have

$$\tilde{F}_{ru}(v) = F(ru, v) = F(u, r^*v) = \tilde{F}_u(r^*v) = (r\tilde{F}_u)(v)$$

and

$$\tilde{F}_{Xu}(v) = F(Xu, v) = \sigma(F(u, Yv)) = \sigma(\tilde{F}_u(Yv)) = (X\tilde{F}_u)(v).$$

Similarly, $\tilde{F}_{Yu} = Y\tilde{F}_u$ for any $u \in V$. Thus \tilde{F} is an A -module homomorphism. \square

The following proposition is analogous the corresponding result proved in [MT1] for finite-dimensional modules over algebras.

Proposition 3.18. *The map $F \mapsto \tilde{F}$ is an isomorphism of abelian groups between the space of admissible forms on V and $\text{Hom}_A(V, V^\sharp)$. Moreover, non-degenerate forms correspond to isomorphisms.*

Proof. Given $\Phi \in \text{Hom}_A(V, V^\sharp)$, define $\hat{\Phi} : V \times V \rightarrow R$ by $\hat{\Phi}(v, w) = \Phi(v)(w)$. Then $\hat{\Phi}$ is an admissible form on V and the maps $F \mapsto \tilde{F}$ and $\Phi \mapsto \hat{\Phi}$ are inverses to each other. If $\hat{\Phi}(v, w) = 0 \forall w$ implies that $v = 0$, then Φ is injective. If $\hat{\Phi}(v, w) = 0 \forall v$ implies that $w = 0$, then Φ is surjective. This proves the last claim. \square

3.8 A semi-simplicity condition

Proposition 3.19. *Let V be a weight A -module, with $\text{Supp}(V)$ contained in a real orbit, such that $\dim_{R/\mathfrak{m}} V_{\mathfrak{m}} = 1 \forall \mathfrak{m} \in \text{Supp}(V)$. If $V^\sharp \simeq V$ then V is semi-simple.*

Proof. If $V^\sharp \simeq V$, then, by Proposition 3.18, V has a non-degenerate admissible form F . Let U be any submodule of V . Then U is itself a weight module and, since $\dim_{R/\mathfrak{m}} V_{\mathfrak{m}} = 1$ for all $\mathfrak{m} \in \text{Supp}(V)$, we have $U = \bigoplus_{\mathfrak{m} \in S} V_{\mathfrak{m}}$ for some subset $S \subseteq \text{Supp}(V)$. Let $U^\perp = \{v \in V : F(v, u) = 0 \forall u \in U\}$. By the defining properties of an admissible form (3.1), U^\perp is an A -submodule of V . On the other hand, by Corollary 3.9 and the non-degeneracy of F , we have $F(V_{\mathfrak{m}}, V_{\mathfrak{n}}) = 0$ iff $\mathfrak{m} \neq \mathfrak{n}$ for $\mathfrak{m}, \mathfrak{n} \in \text{Supp}(V)$. Thus $U^\perp = \bigoplus_{\mathfrak{m} \in \text{Supp}(V) \setminus S} V_{\mathfrak{m}}$. This proves that $U \oplus U^\perp = V$. Hence, any submodule has an invariant complement so V is semi-simple. \square

3.9 Symmetric forms

Recall that the map $R_\omega \rightarrow R_{\omega^*}$ induced by $* : R \rightarrow R$ is denoted by conjugation.

Definition 3.20. Let ω be a symmetric orbit and F an admissible form on a weight A -module V with $\text{Supp}(V) \subseteq \omega$. The *adjoint form* $F^\sharp : V \times V \rightarrow R_\omega$ of F is defined by

$$F^\sharp(v, w) = \overline{F(w, v)}, \quad v, w \in V. \quad (3.9)$$

It is easy to check that F^\sharp is also an admissible form on V . If $F = F^\sharp$, then F is called *symmetric*.

If ω is torsion trivial, we call an admissible \mathbb{K}_ω -form F symmetric if the corresponding admissible form is symmetric.

Proposition 3.21. *Suppose that $\omega \in \Omega$ is symmetric and torsion trivial. Fix $\mathfrak{m} \in \omega$ and put $\mathbb{K}_\omega = R/\mathfrak{m}$. Assume that conjugation on \mathbb{K}_ω is non-trivial, and that the fixed field under conjugation of \mathbb{K}_ω is infinite, of characteristic not two.*

Let V be a finite-dimensional weight A -module with support in ω . If V has a non-degenerate admissible \mathbb{K}_ω -form, then it has a symmetric non-degenerate admissible \mathbb{K}_ω -form.

The proof is exactly as in [MT1], but we provide it for convenience.

Proof. Let $F : V \times V \rightarrow \mathbb{K}_\omega$ be a non-degenerate admissible \mathbb{K}_ω -form on V . Since conjugation is nontrivial, there is an $s \in \mathbb{K}_\omega$ with $\bar{s} = -s$. Then $F_1 = F + F^\sharp$ and $F_2 = s(F - F^\sharp)$ are both symmetric admissible \mathbb{K}_ω -forms. Define $f \in \mathbb{K}_\omega[x]$ by $f(x) = \det(F'_1 + xF'_2)$. Here F'_i denotes the matrix of F_i relative some \mathbb{K}_ω -linear basis of V . Since $f(s^{-1}) = \det(2F') \neq 0$, f is a nonzero polynomial. Among the infinitely many $r \in \mathbb{K}_\omega$ with $\bar{r} = r$, pick one which is not a zero of f . Then $F_1 + rF_2$ is a symmetric non-degenerate admissible \mathbb{K}_ω -form on V . \square

Remark 3.22. Assume R is a finitely generated algebra over an algebraically closed field \mathbb{K} of characteristic zero and assume that σ is a \mathbb{K} -automorphism of R . Let V be an indecomposable weight module over A with support in a real orbit ω . Call two \mathbb{K} -forms F_1, F_2 on V are equivalent if there is an automorphism φ of V and an element $\lambda \in \mathbb{K}, \lambda \neq 0$ such that $F_1(v, w) = \lambda F_2((\varphi(v), \varphi(w)))$ for all $v, w \in V$.

The following statements follow directly from Theorems 2,4 in [MT1].

- 1) If V is simple and $V \simeq V^\sharp$, then there is a unique up to equivalence non-degenerate admissible \mathbb{K} -form on V . If conjugation is nontrivial on \mathbb{K} this form can be chosen to be symmetric, and if conjugation is trivial on \mathbb{K} , the form can be chosen to be symmetric or skew-symmetric.
- 2) If there is a symmetric non-degenerate admissible \mathbb{K} -form on V , then it is unique up to equivalence.

4 The classification of weight modules

In this section we review the classification of indecomposable weight modules with finite-dimensional weight spaces over a generalized Weyl algebra, obtained by Drozd, Guzner and Ovsienko in [DGO].

4.1 Notation

A maximal ideal \mathfrak{m} of R is called a *break* if $t \in \mathfrak{m}$. For $\omega \in \Omega$, let B_ω be the set of all breaks in ω : $B_\omega = \{\mathfrak{m} \in \omega : t \in \mathfrak{m}\}$. Often we put $p = |\omega|$, $m = |B_\omega|$. Let $\mathbb{K}_\mathfrak{m} = R/\mathfrak{m}$. For $r \in R$ we define $r_\mathfrak{m} = r + \mathfrak{m} \in \mathbb{K}_\mathfrak{m}$. For each $\omega \in \Omega$, fix an $\mathfrak{m}(\omega) \in \omega$ and put $\mathbb{K}_\omega = \mathbb{K}_{\mathfrak{m}(\omega)}$.

If $\omega \in \Omega$ is infinite, it is naturally ordered by defining $\mathfrak{m} < \mathfrak{n}$ iff $\mathfrak{n} = \sigma^k(\mathfrak{m})$ for some $k > 0$.

If $|\omega| = p < \infty$, define a ternary relation on ω by $\mathfrak{m} < \mathfrak{m}' < \mathfrak{m}''$ if $\mathfrak{m}' = \sigma^i(\mathfrak{m}), \mathfrak{m}'' = \sigma^k(\mathfrak{m})$ for some $0 < i < k < p$. Let $m = |B_\omega|$ and define a bijective correspondence $\mathbb{Z}_m \rightarrow B_\omega, i \mapsto \mathfrak{m}_i$ such that $i < j < k$ in \mathbb{Z}_m implies $\mathfrak{m}_i < \mathfrak{m}_j < \mathfrak{m}_k$ in ω and $\mathfrak{m}_0 = \mathfrak{m}(\omega)$. For $\mathfrak{m} \in \omega$, let $j(\mathfrak{m})$ denote the only $j \in \mathbb{Z}_m$ such that $\mathfrak{m}_{j-1} < \mathfrak{m} \leq \mathfrak{m}_j$. Let $p_1, p_2, \dots, p_m \in \mathbb{Z}_{>0}$ be minimal such that $\sigma^{p_j}(\mathfrak{m}_{j-1}) = \mathfrak{m}_j$. Equivalently, p_i is the number of $\mathfrak{m} \in \omega$ with $j(\mathfrak{m}) = i$.

Note that $p_1 + p_2 + \cdots + p_m = p$. Furthermore, we put $\tau = \tau_\omega = \sigma^p$. Let $\mathbb{K}_\omega[x, x^{-1}; \tau]$ be the skew Laurent polynomial ring over \mathbb{K}_ω with automorphism τ : $xa = \tau(a)x$ for $a \in \mathbb{K}_\omega$. Similarly, $\mathbb{K}_\omega[x; \tau^k]$ is the skew polynomial ring over \mathbb{K}_ω with automorphism τ^k ($k \in \mathbb{Z}_{\geq 0}$). An element f of such a skew (Laurent) polynomial ring P is called *indecomposable* if the left P -module P/Pf is indecomposable. Two elements $f, g \in P$ are called *similar* if $P/Pf \simeq P/Pg$ as left P -modules.

Let \mathbf{D} denote the free monoid on two letters x, y . Thus \mathbf{D} is the set of words $w = z_1 z_2 \cdots z_n$, where $z_i \in \{x, y\}$, with associative multiplication given by concatenation, and neutral element being the empty word ε of zero length. A word w is an m -word if its length n is a multiple of $m \in \mathbb{Z}_{>0}$. An m -word is *non-periodic* if it is not a power of another m -word. We will let $\sharp : \mathbf{D} \rightarrow \mathbf{D}$, $w \mapsto w^\sharp$, denote the automorphism given by $x^\sharp = y$, $y^\sharp = x$. We also equip \mathbf{D} with a \mathbb{Z} -action given by

$$1.z_1 z_2 \cdots z_n = z_2 z_3 \cdots z_n z_1.$$

for $z_1 z_2 \cdots z_n \in \mathbf{D}$. Following [DGO], we use the notation $w(k)$ for $k.w$.

When ω is symmetric, we will denote the map $\mathbb{K}_\omega \rightarrow \mathbb{K}_\omega$, which is induced by the involution $*$ on R , by conjugation $a \mapsto \bar{a}$.

4.2 The different kinds of modules

4.2.1 Infinite orbit without breaks

Define $V(\omega)$, where $\omega \in \Omega$, $|\omega| = \infty$ and $B_\omega = \emptyset$, as the space $V(\omega) = \oplus_{\mathbf{m} \in \omega} \mathbb{K}_{\mathbf{m}}$ with A -module structure given by $Xv = \sigma(t_{\mathbf{m}}v)$ and $Yv = \sigma^{-1}(v)$ for $v \in \mathbb{K}_{\mathbf{m}}$.

4.2.2 Infinite orbit with breaks

We use an alternative parametrization of these modules, which is more convenient for our purposes. It is easily seen to be equivalent to that of [DGO]. First we need some terminology. Recall the order on infinite orbits ω defined in Section 4.1. An interval S in an infinite orbit ω will be called *supportive* if it satisfies the following property: if S contains a minimal element \mathbf{n}_0 , then $\sigma^{-1}(\mathbf{n}_0) \in B_\omega$ and if S has a maximal element \mathbf{n}_1 , then $\mathbf{n}_1 \in B_\omega$. Let $I(S)$ be the set of *inner breaks* of S :

$$I(S) = \{\mathbf{m} \in S \cap B_\omega : \sigma(\mathbf{m}) \in S\}.$$

Now let $\omega \in \Omega$ be infinite with $B_\omega \neq \emptyset$. Let $S \subseteq \omega$ be a supportive interval and let I_X be any subset of $I(S)$. Define $V(\omega, S, I_X) = \oplus_{\mathbf{m} \in S} \mathbb{K}_{\mathbf{m}}$ with, for $v \in \mathbb{K}_{\mathbf{m}}$,

$$Xv = \begin{cases} \sigma(t_{\mathbf{m}}v), & \text{if } \mathbf{m} \notin B_\omega, \\ \sigma(v), & \text{if } \mathbf{m} \in I_X, \\ 0, & \text{otherwise,} \end{cases} \quad Yv = \begin{cases} \sigma^{-1}(v), & \text{if } \sigma^{-1}(\mathbf{m}) \notin B_\omega, \\ \sigma^{-1}(v), & \text{if } \sigma^{-1}(\mathbf{m}) \in I(S) \setminus I_X, \\ 0, & \text{otherwise.} \end{cases} \quad (4.1)$$

Note that if $V = V(\omega, S, I_X)$ then $S = \text{Supp}(V)$ and $I_X = \{\mathbf{m} \in I(S) : XV_{\mathbf{m}} \neq 0\}$.

4.2.3 Finite orbit without breaks

Given an orbit ω , with $|\omega| = p < \infty$ and $B_\omega = \emptyset$, and an indecomposable polynomial $f = \alpha_0 + \alpha_1 x + \cdots + \alpha_d x^d \in \mathbb{K}_\omega[x, x^{-1}; \tau]$ with $\alpha_0 \neq 0 \neq \alpha_d$, define $V(\omega, f) = \oplus_{\mathbf{m} \in \omega} (\mathbb{K}_{\mathbf{m}})^d$ with A -module structure given by defining for $v \in (\mathbb{K}_{\mathbf{m}})^d$

$$Xv = \begin{cases} \sigma(t_{\mathbf{m}}v), & \text{if } \mathbf{m} \neq \mathbf{m}(\omega), \\ \sigma(F_f t_{\mathbf{m}}v), & \text{if } \mathbf{m} = \mathbf{m}(\omega), \end{cases} \quad (4.2a)$$

$$Yv = \begin{cases} \sigma^{-1}(v), & \text{if } \sigma^{-1}(\mathbf{m}) \neq \mathbf{m}(\omega), \\ F_f^{-1} \sigma^{-1}(v), & \text{if } \sigma^{-1}(\mathbf{m}) = \mathbf{m}(\omega), \end{cases} \quad (4.2b)$$

where

$$F_f = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -\alpha_0/\alpha_d \\ 1 & 0 & 0 & \cdots & 0 & -\alpha_1/\alpha_d \\ 0 & 1 & 0 & \cdots & 0 & -\alpha_2/\alpha_d \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\alpha_{d-1}/\alpha_d \end{bmatrix}.$$

4.2.4 Finite orbit with breaks, first kind

Let $\omega \in \Omega$, $|\omega| = p < \infty$ and $B_\omega \neq \emptyset$. Let $i \in \mathbb{Z}_m$ and $w = z_1 z_2 \cdots z_n \in \mathbf{D}$. Consider $n+1$ symbols e_0, e_1, \dots, e_n . For $\mathbf{m} \in \omega$, let $V_{\mathbf{m}}$ be the vector space over $\mathbb{K}_{\mathbf{m}}$ with basis consisting of all pairs $[\mathbf{m}, e_k]$ such that $i+k = j(\mathbf{m})$ in \mathbb{Z}_m . Put $V(\omega, i, w) = \oplus_{\mathbf{m} \in \omega} V_{\mathbf{m}}$ and supply it with A -module structure by

$$X[\mathbf{m}, e_k] = \begin{cases} \sigma(t_{\mathbf{m}})[\sigma(\mathbf{m}), e_k], & \text{if } \mathbf{m} \notin B_\omega, \\ [\sigma(\mathbf{m}), e_{k+1}], & \text{if } \mathbf{m} \in B_\omega \text{ and } z_{k+1} = x, \\ 0, & \text{otherwise,} \end{cases}$$

$$Y[\mathbf{m}, e_k] = \begin{cases} [\sigma^{-1}(\mathbf{m}), e_k], & \text{if } \sigma^{-1}(\mathbf{m}) \notin B_\omega, \\ [\sigma^{-1}(\mathbf{m}), e_{k-1}], & \text{if } \sigma^{-1}(\mathbf{m}) \in B_\omega \text{ and } z_k = y, \\ 0, & \text{otherwise.} \end{cases}$$

4.2.5 Finite orbit with breaks, second kind

Define $V(\omega, w, f)$, where $\omega \in \Omega$, $|\omega| = p < \infty$ and $|B_\omega| = m > 0$, $w = z_1 z_2 \cdots z_n \in \mathbf{D} \setminus \{\varepsilon\}$ is a non-periodic m -word, and $f = a_1 + a_2 x + \cdots + a_d x^{d-1} + x^d \neq x^d$ is an indecomposable element of $\mathbb{K}_\omega[x; \tau^{n/m}]$ (it should be $\tau^{n/m}$ and not just τ as stated in [DGO]), as follows. Consider dn symbols e_{ks} ($k = 1, \dots, n$, $s = 1, \dots, d$). For $\mathbf{m} \in \omega$, let $V_{\mathbf{m}}$ be the vector space over $\mathbb{K}_{\mathbf{m}}$ with basis consisting of all pairs $[\mathbf{m}, e_{ks}]$ such that $k \equiv j(\mathbf{m}) \pmod{m}$. Define $V(\omega, w, f) = \oplus_{\mathbf{m} \in \omega} V_{\mathbf{m}}$ and supply it with A -module structure by

$$X[\mathfrak{m}, e_{ks}] = \begin{cases} \sigma(t_{\mathfrak{m}})[\sigma(\mathfrak{m}), e_{ks}], & \text{if } \mathfrak{m} \notin B_{\omega}, \\ [\sigma(\mathfrak{m}), e_{k+1,s}], & \text{if } \mathfrak{m} \in B_{\omega}, k < n, z_{k+1} = x, \\ [\sigma(\mathfrak{m}), e_{1,s+1}], & \text{if } \mathfrak{m} \in B_{\omega}, k = n, z_1 = x, s < d, \\ -\sum_{r=1}^d \sigma(a_r)[\sigma(\mathfrak{m}), e_{1r}], & \text{if } \mathfrak{m} \in B_{\omega}, k = n, z_1 = x, s = d, \\ 0, & \text{otherwise,} \end{cases} \quad (4.3)$$

$$Y[\mathfrak{m}, e_{ks}] = \begin{cases} [\sigma^{-1}(\mathfrak{m}), e_{ks}], & \text{if } \sigma^{-1}(\mathfrak{m}) \notin B_{\omega}, \\ [\sigma^{-1}(\mathfrak{m}), e_{k-1,s}], & \text{if } \sigma^{-1}(\mathfrak{m}) \in B_{\omega}, k > 1, z_k = y, \\ [\sigma^{-1}(\mathfrak{m}), e_{n,s-1}], & \text{if } \sigma^{-1}(\mathfrak{m}) \in B_{\omega}, k = 1, z_1 = y, s > 1, \\ -\sum_{r=1}^d a_r^{\circ}[\sigma^{-1}(\mathfrak{m}), e_{nr}], & \text{if } \sigma^{-1}(\mathfrak{m}) \in B_{\omega}, k = 1, z_1 = y, s = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (4.4)$$

Here $a_{d+1-r}^{\circ} = \tau^{r-1}(a_r)$, i.e. $a_r^{\circ} = \tau^{d-r}(a_{d+1-r})$. As compared to [DGO], we changed notation from e_{ks} to $e_{k,d+1-s}$ in the case when $z_1 = y$.

The weight diagram of a module of the form $V = V(\omega, w, f)_{\mathfrak{m}}$, where the first letter of w is $z_1 = x$, is illustrated in Figure 1. Each dot \bullet is a one-dimensional (over R/\mathfrak{m}) subspace of the weight space $V_{\mathfrak{m}}$. Arrows going in the right direction correspond to X while left arrows correspond to Y . The diagram $\mathfrak{m} \xrightarrow{\sigma(\mathfrak{m})} \bullet$ means that X and Y act bijectively on the corresponding one-dimensional subspaces. We shall write

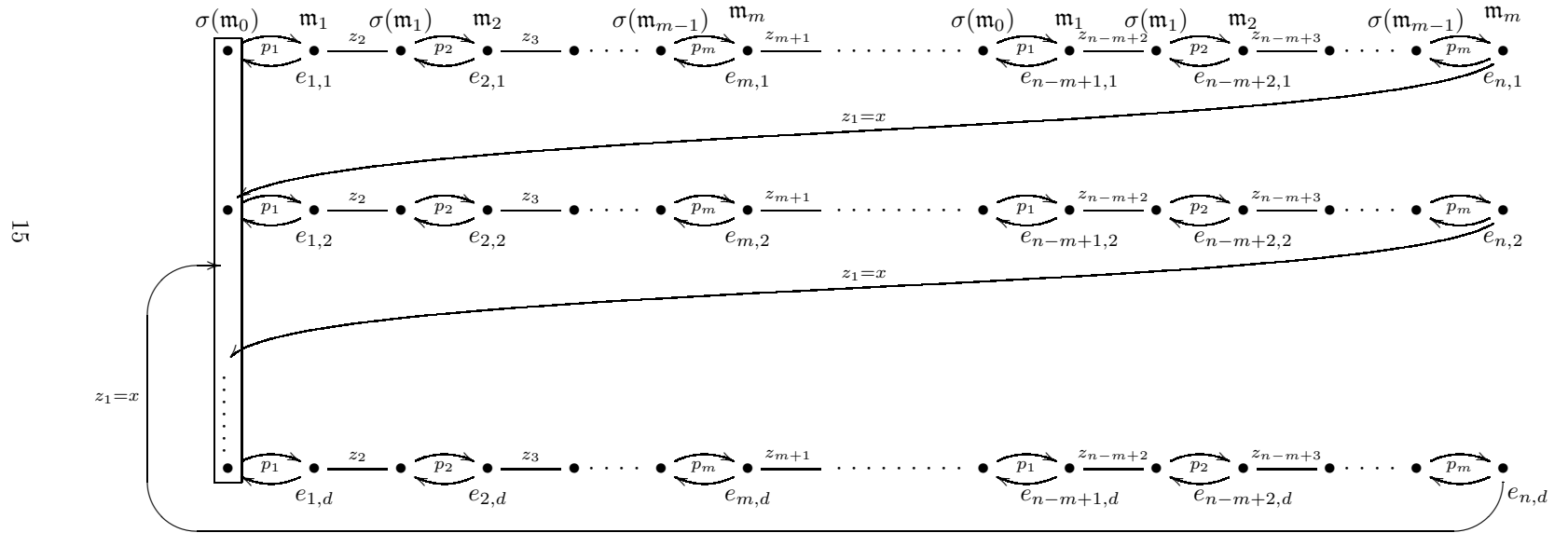
$$\begin{array}{ccc} \sigma(\mathfrak{m}) & \xrightarrow{\sigma^n(\mathfrak{m})} & \\ \bullet & \xrightleftharpoons[n]{n} & \bullet \end{array}$$

to denote the weight diagram

$$\begin{array}{ccccccc} \sigma(\mathfrak{m}) & \xrightarrow{\sigma^2(\mathfrak{m})} & & \xrightarrow{\sigma^{n-1}(\mathfrak{m})} & \sigma^n(\mathfrak{m}) & & \\ \bullet & \xrightleftharpoons{\quad} & \bullet & \xrightleftharpoons{\quad} & \bullet & \cdots & \bullet \end{array}$$

The diagram $\mathfrak{m} \xrightarrow{z} \sigma(\mathfrak{m})$ where $z \in \{x, y\}$, means that if $z = x$ then X acts bijectively from \mathfrak{m} to $\sigma(\mathfrak{m})$ and Y acts as zero on $\sigma(\mathfrak{m})$ while if $z = y$, then Y is bijective as a map from $\sigma(\mathfrak{m})$ to \mathfrak{m} and X acts as zero on \mathfrak{m} . Often, in weight diagrams each weight space is depicted as a column of dots. In Figure 1, however, for clarity, each column is only a subspace of a certain weight space, and each weight is repeated n/m times horizontally. Recall that, by convention, $p_m = p_0$ and $\mathfrak{m}_m = \mathfrak{m}_0$.

Figure 1: Weight diagram for $V(\omega, w, f)$ when $z_1 = x$.



4.3 The classification theorem

Theorem 4.1 ([DGO], Theorem 5.7).

- (i) The A -modules $V(\omega), V(\omega, f), V(\omega, S, I_X), V(\omega, i, w)$, and $V(\omega, w, f)$ are indecomposable weight A -modules.
- (ii) Every weight A -module V such that $\dim_{\mathbb{K}_m} V_m < \infty$ whenever m belong to a finite orbit, decomposes uniquely into a direct sum of modules isomorphic to those listed in (i).
- (iii) The only isomorphisms between the listed modules are the following:

- If f and g are similar in $\mathbb{K}_\omega[x, x^{-1}; \tau]$, then

$$V(\omega, f) \simeq V(\omega, g). \quad (4.5)$$

- If f and g are similar in $\mathbb{K}_\omega[x; \tau^{n/m}]$, and $i \in \mathbb{Z}$, then

$$V(\omega, w, f) \simeq V(\omega, w(mi), \tau^i(g)), \quad (4.6)$$

where $m = |B_\omega|$ and $n = |w|$.

Remark 4.2. In [DGO], τ^i is uncorrectly missing from (4.6). In general, if i is not a multiple of n/m , then f is not similar to $\tau^i(f)$ in $\mathbb{K}_\omega[x; \tau^{n/m}]$. But for $g = f$, one can construct an isomorphism $\varphi : V(\omega, w(m), \tau(f)) \rightarrow V(\omega, w, f)$ determined by the conditions

$$1) \quad \varphi([\sigma(m_0), e_{1,1}]) = [\sigma(m_0), e_{m+1,1}], \quad (4.7)$$

$$2) \quad \varphi([m, e_{k,s}]) \in \begin{cases} \oplus_{r=1}^d \mathbb{K}_m[m, e_{k+m,r}] & k+m \leq n, \\ \oplus_{r=1}^d \mathbb{K}_m[m, e_{k+m-n,r}] & k+m > n. \end{cases} \quad (4.8)$$

Remark 4.3. Taking $i = n/m$ in (4.6) we deduce that f is similar $\tau^{n/m}(f)$ in $P := \mathbb{K}_\omega[x; \tau^{n/m}]$. This isomorphism is explicitly given by

$$\begin{aligned} \varphi : P/P\tau^{n/m}(f) &\rightarrow P/Pf \\ g + P\tau^{n/m}(f) &\mapsto gx + Pf. \end{aligned}$$

This map is well defined since $\tau^{n/m}(f)x = xf$. It is a homomorphism of left P -modules. Moreover, since $f \neq x^d$ and is indecomposable, its constant term is nonzero. Therefore φ is surjective. Since dimensions agree, φ is an isomorphism as claimed.

The following description of the simple weight A -modules was also given.

Theorem 4.4 ([DGO], Theorem 5.8). *The weight A -modules $V(\omega), V(\omega, f)$ for irreducible $f \in \mathbb{K}_\omega[x, x^{-1}; \tau]$, $V(\omega, S, \emptyset)$ for supportive interval $S \subseteq \omega$ with $I(S) = \emptyset$, $V(\omega, i, \varepsilon)$ and $V(\omega, w, f)$ for irreducible $f \in \mathbb{K}_\omega[x; \tau^{n/m}]$ and $w = x^m$ or $w = y^m$ where $m = |B_\omega|$, are simple and each simple weight A -module is isomorphic to one from this list.*

5 Description of indecomposable weight modules having a non-degenerate admissible form

In this section we consider in turn each of the five types of indecomposable modules from the DGO classification in Section 4 and determine necessary and sufficient conditions, in terms of the parameters, for the modules to be isomorphic to their finitistic dual which, by Proposition 3.18, is equivalent to having a non-degenerate admissible form. We will only consider the case when $\text{Supp}(V)$ is contained in a real orbit ω . The case of symmetric nonreal orbit will be left for future studies.

The following lemma will be useful.

Lemma 5.1. *If V is indecomposable, then so is V^\sharp .*

Proof. We prove that if V is decomposable, then so is V^\sharp . Then the result follows since $V^\sharp \simeq V$, by Proposition 3.15. Assume V is decomposable and let $i_j : U_j \rightarrow V$, $j = 1, 2$, be the inclusions of two submodules U_j whose direct sum is V . Let $W_j = \ker(i_j^\sharp) \subseteq V^\sharp$, $j = 1, 2$. Let $\varphi \in W_1 \cap W_2$. Then $i_1^\sharp(\varphi) = 0 = i_2^\sharp(\varphi)$. Thus $\varphi(i_j(u)) = 0 \forall u \in U_j$, $j = 1, 2$. Since $V = i_1(U_1) + i_2(U_2)$ we deduce $\varphi = 0$. Hence $W_1 \cap W_2 = 0$. Let $\varphi \in V^\sharp$ be arbitrary. Then $\varphi p_1 + \varphi p_2 = \varphi$, where $p_j : V \rightarrow U_j$ are the projections. Also $i_1^\sharp(\varphi p_2)(v) = (\varphi p_2)(i_1(v)) = 0 \forall v \in U_1$, and similarly $i_2^\sharp(\varphi p_1) = 0$. This proves that $V^\sharp = W_1 + W_2$. \square

5.1 Infinite orbit without breaks

Theorem 5.2. *Let $V = V(\omega)$, where ω is an infinite real orbit with $B_\omega = \emptyset$. Then $V^\sharp \simeq V$.*

Proof. We have $\text{Supp}(V) = \omega$. By the classification theorem, there is only one indecomposable module whose support is contained in ω . By Lemma 5.1, V^\sharp is indecomposable and by Proposition 3.14, $\text{Supp}(V^\sharp) = \text{Supp}(V) = \omega$. Hence we conclude that $V^\sharp \simeq V$. \square

Let ω be infinite real, $B_\omega = \emptyset$, $V = V(\omega)$. We now determine all non-degenerate admissible forms on V , and their index in the symmetric complex case. Let $e_0 \in V_{\mathfrak{m}(\omega)}$, $e_0 \neq 0$. Let $e_0^\sharp \in V^\sharp$ be defined by $e_0^\sharp(e_0) = 1_{\mathfrak{m}(\omega)}$ and $e_0^\sharp(V_{\mathfrak{m}}) = 0 \forall \mathfrak{m} \in \omega, \mathfrak{m} \neq \mathfrak{m}(\omega)$. Then e_0^\sharp spans $(V^\sharp)_{\mathfrak{m}(\omega)}$ over \mathbb{K}_ω so any isomorphism $\Phi : V \rightarrow V^\sharp$ must satisfy $\Phi(e_0) = \lambda e_0^\sharp$ for some nonzero $\lambda \in \mathbb{K}_\omega$. Conversely, it is easy to see that for any nonzero $\lambda \in \mathbb{K}_\omega$ there exists a unique isomorphism $\Phi_\lambda : V \rightarrow V^\sharp$ satisfying $\Phi_\lambda(e_0) = \lambda e_0^\sharp$. The set $\{e_n := X^n e_0, e_{-n-1} := Y^{n+1} e_0 \mid n \in \mathbb{Z}_{\geq 0}\}$ is a basis for V over \mathbb{K}_ω and the corresponding \mathbb{K}_ω -form Ψ_λ (which is obtained using the bijections between $\text{Hom}_A(V, V^\sharp)$ and admissible forms in Proposition 3.18 and between admissible

forms and \mathbb{K}_ω -forms in Proposition 3.4) satisfies

$$\begin{aligned}\Psi_\lambda(e_n, e_m) &= 0, \quad m \neq n, \\ \Psi_\lambda(e_n, e_n) &= \begin{cases} t\sigma^{-1}(t) \cdots \sigma^{-n+1}(t)\lambda, & n \geq 0, \\ \sigma(t)\sigma^2(t) \cdots \sigma^{-n}(t)\lambda, & n < 0. \end{cases}\end{aligned}\quad (5.1)$$

To simplify notation we use here the natural R -module action on \mathbb{K}_ω . For example $t\lambda$ equals the product $(t + \mathfrak{m}(\omega))\lambda$ in \mathbb{K}_ω . From the formula (5.1), and that $t^* = t$, we see that the adjoint form Ψ_λ^\sharp is equal to $\Psi_{\bar{\lambda}}$.

In the case when $\mathbb{K}_\omega \simeq \mathbb{C}$ and conjugation is ordinary complex conjugation, we associate to a symmetric form Ψ_λ , $\lambda \in \mathbb{R}$, a scalar product on V defined by $(e_k, e_l) = \text{sgn}(\Psi_\lambda(e_k, e_l))\Psi_\lambda(e_k, e_l)$. Then $\Psi_\lambda(v, w) = (Jv, w) \forall v, w \in V$, where $Je_k = \text{sgn}(\Psi_\lambda(e_k, e_k))e_k$. J is an involution operator in the sense that $J^2 = \text{Id}_V$ and that it is self-adjoint with respect to the scalar product on V . Therefore, (the completion of) V together with Ψ_λ is a Krein space (see [KS]). Let $V_\pm = \{v \in V : Jv = \pm v\}$. Then $V = V_+ \oplus V_-$. We claim that any pair $(\dim V_+, \dim V_-)$ can occur. In fact, consider the sequence $(i_n)_{n \in \mathbb{Z}}$ where $i_n = \text{sgn}(\Psi_\lambda(e_n, e_n))$. Then any sequence $(i_n)_{n \in \mathbb{Z}} \in \{1, -1\}^{\mathbb{Z}}$ can occur. Indeed, let $R = \mathbb{C}[t_n \mid n \in \mathbb{Z}]$ be a polynomial algebra in infinitely many indeterminates t_n . Let $t = t_0$, define $t_n^* = t_n$, $i^* = -i$ and extend $*$ to an \mathbb{R} -algebra automorphism of R . Let $\sigma(t_n) = t_{n+1}$ and let \mathfrak{m} be the maximal ideal generated by $t_n - a_n$, $n \in \mathbb{Z}$, where $a_n \in \mathbb{R}$ are given by $a_n = i_{-n}i_{-n+1}$, $n \in \mathbb{Z}$. Let ω be the orbit containing \mathfrak{m} and set $\mathfrak{m}(\omega) = \mathfrak{m}$. The orbit ω is infinite, real, and $B_\omega = \emptyset$. Then the sequence associated to the form Ψ_{i_0} on $V(\omega)$ equals $(i_n)_{n \in \mathbb{Z}}$.

5.2 Infinite orbit with breaks

Theorem 5.3. *Let $V = V(\omega, S, I_X)$, where $\omega \in \Omega$ is infinite and real, $|B_\omega| > 0$, $S \subseteq \omega$ is a supportive interval, and $I_X \subseteq I(S)$. Then $V^\sharp \simeq V(\omega, S, I(S) \setminus I_X)$. In particular V has a non-degenerate admissible form iff $I(S) = \emptyset$ which is equivalent to V being simple.*

Proof. If $V^\sharp \simeq V$, then Proposition 3.19 and that V is indecomposable imply that V must be simple. The converse follows when we prove the more general statement that $V^\sharp \simeq V(\omega, S, I(S) \setminus I_X)$.

By Lemma 5.1, V^\sharp is indecomposable and by Proposition 3.14 and that ω is real, $\text{Supp}(V^\sharp) = \text{Supp}(V) = S$. So by the classification theorem, Theorem 4.1, we deduce that $V^\sharp \simeq V(\omega, S, J)$ for some subset J of $I(S)$. It remains to prove that, for $\mathfrak{m} \in I(S)$, $X(V^\sharp)_\mathfrak{m} \neq 0$ iff $XV_\mathfrak{m} = 0$.

Suppose $\mathfrak{m} \in I(S)$ with $X(V^\sharp)_\mathfrak{m} = 0$. Let $\varphi \in (V^\sharp)_\mathfrak{m}$ be nonzero. Then, by Proposition 3.13, $\varphi|_{V_\mathfrak{n}} = 0$ if $\mathfrak{n} \neq \mathfrak{m}$ and $\varphi(v) = 1_\mathfrak{m}$ for some $v \in V_\mathfrak{m}$. Let $u \in V_{\sigma(\mathfrak{m})}$ be nonzero. We have $0 = (X\varphi)(u) = \sigma(\varphi(Yu))$. Thus $Yu = 0$. Thus $u = Xv$ for some nonzero $v \in V_\mathfrak{m}$, otherwise V would be decomposable into $(\oplus_{n \leq 0} V_{\sigma^n(\mathfrak{m})}) \oplus (\oplus_{n > 0} V_{\sigma^n(\mathfrak{m})})$. This proves that $\mathfrak{m} \in I_X$, i.e. $XV_\mathfrak{m} \neq 0$. The converse is similar.

We conclude that indeed $V^\sharp \simeq V$ iff $I(S) = \emptyset$. By Theorem 4.4, $V(\omega, S, I_X)$ is simple iff $I(S) = \emptyset$. \square

Let $\omega \in \Omega$ be real, infinite, $|B_\omega| > 0$. In this case ω is torsion trivial and thus there is a bijection between admissible forms and admissible \mathbb{K}_ω -forms. We now determine all possible non-degenerate admissible \mathbb{K}_ω -forms on $V(\omega, S, \emptyset)$ where S is a supportive interval in ω with $I(S) = \emptyset$.

The subset $S \subseteq \omega$ has either a maximal or a minimal element (otherwise it would contain an inner break). Assume S has a maximal element \mathbf{n}_1 . It is a break since S is supportive. We can assume that $\mathbf{m}(\omega) = \mathbf{n}_1$. Let $e_0 \in V_{\mathbf{m}(\omega)}$ be nonzero and $e_0^\sharp \in (V^\sharp)_{\mathbf{m}(\omega)}$ be such that $e_0^\sharp(e_0) = 1_{\mathbf{m}(\omega)}$. For $\lambda \in \mathbb{K}_\omega$ there is a unique isomorphism $\Phi_\lambda : V \rightarrow V^\sharp$ given by $\Phi_\lambda(e_0) = \lambda e_0^\sharp$. If S has no minimal element, V has a basis $\{e_{-n} := Y^n e_0 \mid n \geq 0\}$. If S has a minimal element \mathbf{n}_0 , then $\sigma^{-1}(\mathbf{n}_0) \in B_\omega$ and V has a basis $\{e_{-n} := Y^n e_0 \mid 0 \leq n \leq N-1\}$ where $\sigma^{-N}(\mathbf{m}(\omega)) = \sigma^{-1}(\mathbf{n}_0)$. The corresponding \mathbb{K}_ω -form Ψ_λ calculated on the basis vectors gives

$$\Phi_\lambda(e_{-n}, e_{-m}) = \sigma(t)\sigma^2(t) \cdots \sigma^n(t)\lambda\delta_{n,m} \quad (5.2)$$

for $n, m \geq 0$. If S has no maximal element, but a minimal element \mathbf{n}_0 , then $\sigma^{-1}(\mathbf{n}_0) \in B_\omega$. We choose $\mathbf{m}(\omega) = \mathbf{n}_0$ in this case. Then V has a basis $\{e_n := X^n e_0 \mid n \geq 0\}$ and the corresponding \mathbb{K}_ω -form Ψ_λ satisfies

$$\Psi_\lambda(e_n, e_m) = t\sigma^{-1}(t) \cdots \sigma^{-n+1}(t)\lambda\delta_{n,m} \quad (5.3)$$

for $n, m \geq 0$. We see that Ψ_λ is symmetric iff $\bar{\lambda} = \lambda$.

5.3 Finite orbit without breaks

In this section we fix a finite orbit $\omega \in \Omega$ with $B_\omega = \emptyset$. In Theorem 5.6 we will describe the dual modules $V(\omega, f)^\sharp$ for indecomposable $f \in \mathbb{K}_\omega[x, x^{-1}; \tau]$. First we make some preliminary observations. Let $p = |\omega|$ and put $P = \mathbb{K}_\omega[x, x^{-1}; \tau]$.

Proposition 5.4. *Let B be the subalgebra of A generated by X^p, Y^p and all $r \in R$. Let $I = B\mathbf{m}(\omega)B$ be the ideal in B generated by $\mathbf{m}(\omega)$. Then there is a ring isomorphism*

$$\psi : B/I \rightarrow P$$

given by

$$\psi(X^p + I) = \xi \cdot x, \quad \psi(Y^p + I) = x^{-1}, \quad \psi(r + I) = r_{\mathbf{m}(\omega)} \quad \text{for } r \in R,$$

where

$$\xi = (\sigma(t)\sigma^2(t) \cdots \sigma^p(t))_{\mathbf{m}(\omega)}. \quad (5.4)$$

Proof. The map ψ is a well-defined ring homomorphism, using the relations (2.1) in A . Assume $b + I \in B/I$ is in the kernel of ψ . Since both rings involved, and ψ , are \mathbb{Z} -graded in a natural way, we can assume $b = rX^{pk}$ or $b = rY^{pk}$, $k \geq 0$. We immediately get $k = 0$, $r \in \mathbf{m}(\omega)$. So ψ is injective. That ψ is surjective is easy to see. \square

Let $V = V(\omega, f)$, where $f = \alpha_0 + \alpha_1 x + \cdots + \alpha_d x^d \in P$, ($\alpha_0 \neq 0, \alpha_d \neq 0$), is indecomposable. Since ω is an orbit of length p , we have $BV_{\mathfrak{m}(\omega)} \subseteq V_{\mathfrak{m}(\omega)}$. Also $IV_{\mathfrak{m}(\omega)} = 0$. Thus $V_{\mathfrak{m}(\omega)}$ becomes a module over B/I and, via the isomorphism in Proposition 5.4, a P -module. The following proposition describes this P -module.

Proposition 5.5.

$$V_{\mathfrak{m}(\omega)} \simeq P/Pf$$

as P -modules.

Proof. Let $e_i = (0, \dots, \overset{i}{1}, \dots, 0) \in V_{\mathfrak{m}(\omega)} = (\mathbb{K}_\omega)^d$. By (4.2a), if $1 \leq i < d$,

$$\begin{aligned} X^p e_i &= X^{p-1} \sigma(F_f t_{\mathfrak{m}(\omega)} e_i) = \sigma^p(t_{\mathfrak{m}(\omega)}) X^{p-1} \sigma(e_{i+1}) = \\ &= \sigma^p(t_{\mathfrak{m}(\omega)}) \sigma^{p-1}(t_{\sigma(\mathfrak{m}(\omega))}) X^{p-2} \sigma^2(e_{i+1}) = \cdots = \\ &= \xi \cdot e_{i+1}. \end{aligned}$$

Thus

$$(\xi^{-1} X^p)^k e_1 = e_{k+1} \quad \text{for } k = 0, 1, \dots, d-1. \quad (5.5)$$

Also we have, by (4.2a),

$$\xi^{-1} X^p e_d = \sum_{k=0}^{d-1} \tau(-\alpha_k/\alpha_d) e_{k+1}. \quad (5.6)$$

Using (5.5) and (5.6) we get

$$\begin{aligned} \tau(f) \cdot e_1 &= \sum_{k=0}^d \tau(\alpha_k) x^k \cdot e_1 = \sum_{k=0}^d \tau(\alpha_k) (\xi^{-1} X^p)^k e_1 = \\ &= \sum_{k=0}^{d-1} \tau(\alpha_k) e_{k+1} + \tau(\alpha_d) \sum_{k=0}^{d-1} \tau(-\alpha_k/\alpha_d) e_{k+1} = 0. \end{aligned} \quad (5.7)$$

From (5.5) and that $\{e_1, \dots, e_d\}$ generates $V_{\mathfrak{m}(\omega)}$ as an R -module, we see that the vector e_1 generates $V_{\mathfrak{m}(\omega)}$ as a P -module. By (5.7), we get an epimorphism of P -modules

$$\begin{aligned} \psi : P/P\tau(f) &\rightarrow V_{\mathfrak{m}(\omega)} \\ h + P\tau(f) &\mapsto h \cdot e_1 \end{aligned}$$

Since $\dim_{\mathbb{K}_\omega} V_{\mathfrak{m}(\omega)} = d = \dim_{\mathbb{K}_\omega} P/P\tau(f)$, we deduce that ψ is an isomorphism. Since f is similar to $\tau(f)$, it follows that $V_{\mathfrak{m}(\omega)} \simeq P/Pf$. \square

Now we come to the main result in this section.

Theorem 5.6. *Let $V = V(\omega, f)$, where ω is a finite and real orbit with $B_\omega = \emptyset$ and $f = \alpha_0 + \alpha_1 x + \cdots + \alpha_d x^d \in P = \mathbb{K}_\omega[x, x^{-1}; \tau]$, $\alpha_0 \neq 0 \neq \alpha_d$, is indecomposable. Then*

$$V(\omega, f)^\# \simeq V(\omega, f^\#)$$

with

$$f^\# = \sum_{k=0}^d \{k\}_\xi \cdot \tau^k(\overline{\alpha_{d-k}}) \cdot x^k, \quad (5.8)$$

where

$$\{k\}_\xi = \xi \tau(\xi) \cdots \tau^{k-1}(\xi) \quad \text{for } k \geq 0, \quad (5.9)$$

and

$$\xi = (\sigma(t)\sigma^2(t) \cdots \sigma^p(t))_{\mathfrak{m}(\omega)}. \quad (5.10)$$

In particular, $V \simeq V^\#$ iff f is similar to $f^\#$ in P .

Proof. By Lemma 5.1 and Proposition 3.14, $V^\#$ is indecomposable and the support $\text{Supp}(V^\#) = \omega$. So by Theorem 4.1, we know that $V^\# \simeq V(\omega, h)$ for some $h \in P$. Then by Proposition 5.5, $(V^\#)_{\mathfrak{m}(\omega)} \simeq P/Ph$. Thus, it is enough to prove that $(V^\#)_{\mathfrak{m}(\omega)} \simeq P/Pf^\#$ as P -modules, because then h is similar to $f^\#$ which implies that $V^\# \simeq V(\omega, f^\#)$ by the isomorphism (4.5).

For this, let $e_i = (0, \dots, \overset{i}{1}, \dots, 0) \in V_{\mathfrak{m}(\omega)} = (\mathbb{K}_\omega)^d$, and define $e_i^\# \in V^\#$ by $e_i^\#(V_{\mathfrak{n}}) = 0$ for $\mathfrak{n} \in \omega$, $\mathfrak{n} \neq \mathfrak{m}(\omega)$ and $e_i^\#(e_k) = \delta_{ik} \cdot 1_{\mathfrak{m}(\omega)}$ for $i, k = 1, \dots, d$. Since ω is real, $e_i^\# \in (V^\#)_{\mathfrak{m}(\omega)}$. By relation (4.2b),

$$Y^p e_k = \begin{cases} e_{k-1}, & k > 1, \\ F_f^{-1} e_1, & k = 1. \end{cases} \quad (5.11)$$

It is easy to check that

$$F_f^{-1} e_1 = -\alpha_0^{-1}(\alpha_1 e_1 + \alpha_2 e_2 + \cdots \alpha_d e_d). \quad (5.12)$$

Thus for any $i, k = 1, \dots, d$,

$$\begin{aligned} (X^p e_i^\#)(e_k) &= \tau(e_i^\#(Y^p e_k)) = \begin{cases} \delta_{i, k-1} \cdot 1_{\mathfrak{m}(\omega)}, & k > 1, \\ \tau(-\overline{\alpha_i}/\overline{\alpha_0}) \cdot 1_{\mathfrak{m}(\omega)}, & k = 1, \end{cases} \\ &= (e_{i+1}^\# - \tau(\overline{\alpha_i}/\overline{\alpha_0}) \cdot e_1^\#)(e_k) \end{aligned} \quad (5.13)$$

with the convention that $e_i^\# = 0$ for $i > d$. Let also $\alpha_i = 0$ for $i > d$. We claim that

$$\sum_{k=0}^n \tau^{k+1}(\overline{\alpha_{n-k}}/\overline{\alpha_0}) \cdot X^{pk} e_1^\# = e_{n+1}^\#, \quad \text{for all } n \geq 0. \quad (5.14)$$

We prove this by induction on n . For $n = 0$ it is trivial. Assume that

$$\sum_{k=0}^{n-1} \tau^{k+1}(\overline{\alpha_{n-1-k}}/\overline{\alpha_0}) \cdot X^{pk} e_1^\# = e_n^\#$$

Apply X^p to both sides to get

$$\sum_{k=0}^{n-1} \tau^{k+2}(\overline{\alpha_{n-1-k}}/\overline{\alpha_0}) \cdot X^{p(k+1)} e_1^\# = X^p e_n^\#$$

Use that, by (5.13), $X^p e_n^\sharp = e_{n+1}^\sharp - \tau(\overline{\alpha_n}/\overline{\alpha_0}) \cdot e_1^\sharp$ in the right hand side, add $\tau(\overline{\alpha_n}/\overline{\alpha_0}) \cdot e_1^\sharp$ to both sides, and replace k by $k-1$ in the sum in the left hand side to obtain

$$\sum_{k=1}^n \tau^{k+1}(\overline{\alpha_{n-k}}/\overline{\alpha_0}) \cdot X^{pk} e_1^\sharp + \tau(\overline{\alpha_n}/\overline{\alpha_0}) \cdot e_1^\sharp = e_{n+1}^\sharp.$$

This proves (5.14). From (5.14) we see that e_1^\sharp generates $(V^\sharp)_{\mathfrak{m}(\omega)}$ as a P -module and that $g \cdot e_1^\sharp = 0$, where

$$g = \sum_{k=0}^d \tau^{k+1}(\overline{\alpha_{d-k}}/\overline{\alpha_0})(\xi x)^k = \sum_{k=0}^d \xi \tau(\xi) \cdots \tau^{k-1}(\xi) \cdot \tau^{k+1}(\overline{\alpha_{d-k}}/\overline{\alpha_0}) x^k \in P.$$

Thus, as in the proof of Proposition 5.5, $(V^\sharp)_{\mathfrak{m}(\omega)} \simeq P/Pg$ as P -modules. Moreover, one verifies that $\tau^{-1}(\xi) \cdot \tau^{-1}(g) \cdot \tau^{-1}(\xi) \overline{\alpha_0} = f^\sharp$. Thus g is similar to f^\sharp and we conclude that $(V^\sharp)_{\mathfrak{m}(\omega)} \simeq P/Pf^\sharp$. This finishes the proof of the theorem. \square

Remark 5.7. The example in Section 6.3, concerning $U_q(\mathfrak{sl}_2)$, shows that there exist non-simple indecomposable weight modules which are unitarizable with a non-degenerate admissible form. This is in contrast to the case of bounded $*$ -representations of $*$ -algebras on Hilbert spaces, that is, unitarizable modules with respect to a positive definite form, where any unitarizable module is semisimple. The example also shows that not all simple weight modules have a non-degenerate admissible form.

5.4 Finite orbit with breaks, first kind

Recall that we defined an automorphism of order two of the monoid \mathbf{D} by $x^\sharp = y$ and $y^\sharp = x$. For example, $(xy)^\sharp = yx$.

Theorem 5.8. *Let ω be a finite real orbit with $m := |B_\omega| > 0$, let $j \in \mathbb{Z}_m$ and let $w \in \mathbf{D}$. Then $V(\omega, j, w)^\sharp \simeq V(\omega, j, w^\sharp)$. In particular $V(\omega, j, w)$ has a nondegenerate admissible form iff $w = \varepsilon$, the empty word (of length $n = 0$), which is equivalent to that $V(\omega, j, w)$ is simple.*

Proof. Define $\Phi : V(\omega, j, w) \rightarrow V(\omega, j, w^\sharp)^\sharp$ by $\Phi([m, e_k]) = c_{m,k}[m, e_k^\sharp]$ where $[m, e_k^\sharp] \in V(\omega, j, w^\sharp)^\sharp$ are defined by $[m, e_k^\sharp]([n, e_l]) = \delta_{n,m} \delta_{k,l} \cdot 1_m$ (where $1_m = 1 + m \in R/\mathfrak{m} \subseteq R_\omega$) and the coefficients $c_{m,k} \in R/\mathfrak{m}$ are nonzero, to be determined later. Extend Φ to an R -module isomorphism.

Let $[m, e_k]$ be a basis vector of $V(\omega, j, w)$. Thus $j + k \equiv j(\mathfrak{m}) \pmod{m}$. Write $w = z_1 \cdots z_n$. Consider a basis vector of the form $[\sigma(\mathfrak{m}), e_l] \in V(\omega, j, w^\sharp)$.

We have

$$\begin{aligned}
& \left(X\Phi([\mathbf{m}, e_k]) \right) ([\sigma(\mathbf{m}), e_l]) = \sigma \left(c_{\mathbf{m},k} [\mathbf{m}, e_k^\#] (Y[\sigma(\mathbf{m}), e_l]) \right) = \\
& = \begin{cases} \sigma \left(c_{\mathbf{m},k} [\mathbf{m}, e_k^\#] ([\mathbf{m}, e_l]) \right), & \mathbf{m} \notin B_\omega, \\ \sigma \left(c_{\mathbf{m},k} [\mathbf{m}, e_k^\#] ([\mathbf{m}, e_{l-1}]) \right), & \mathbf{m} \in B_\omega \text{ and } z_l^\# = y, \\ 0, & \text{otherwise} \end{cases} \\
& = \begin{cases} \sigma(c_{\mathbf{m},k}) \delta_{kl} \cdot 1_{\sigma(\mathbf{m})}, & \mathbf{m} \notin B_\omega, \\ \sigma(c_{\mathbf{m},k}) \delta_{k,l-1} \cdot 1_{\sigma(\mathbf{m})}, & \mathbf{m} \in B_\omega \text{ and } z_l = x, \\ 0, & \text{otherwise} \end{cases} \\
& = \begin{cases} \sigma(c_{\mathbf{m},k}) c_{\sigma(\mathbf{m}),k}^{-1} \left(\Phi([\sigma(\mathbf{m}), e_k]) \right) ([\sigma(\mathbf{m}), e_l]), & \mathbf{m} \notin B_\omega, \\ \sigma(c_{\mathbf{m},k}) c_{\sigma(\mathbf{m}),k+1}^{-1} \left(\Phi([\sigma(\mathbf{m}), e_{k+1}]) \right) ([\sigma(\mathbf{m}), e_l]), & \mathbf{m} \in B_\omega \text{ and } z_{k+1} = x, \\ 0, & \text{otherwise.} \end{cases} \\
& = \left(\Phi(X[\mathbf{m}, e_k]) \right) ([\sigma(\mathbf{m}), e_l])
\end{aligned}$$

if $c_{\mathbf{m},k}$ are chosen in such a way that $\sigma(c_{\mathbf{m},k})/c_{\sigma(\mathbf{m}),k} = \sigma(t_{\mathbf{m}})$ when $\mathbf{m} \notin B_\omega$ and $\sigma(c_{\mathbf{m},k})/c_{\sigma(\mathbf{m}),k+1} = 1$ when $\mathbf{m} \in B_\omega$ and $z_{k+1} = x$. On other basis vectors $[\mathbf{n}, e_l]$, $\mathbf{n} \neq \sigma(\mathbf{m})$, both sides are zero:

$$\left(X\Phi([\mathbf{m}, e_k]) \right) ([\mathbf{n}, e_l]) = 0 = \left(\Phi(X[\mathbf{m}, e_k]) \right) ([\mathbf{n}, e_l]).$$

With this choice of coefficients, Φ commutes with the action of X . For the action of Y , suppose v is a basis vector of $V(\omega, j, w)$ which is equal to Xu for some u . Then

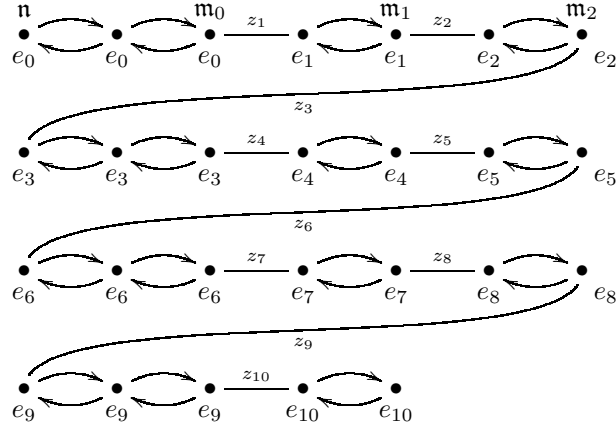
$$\Phi(Yv) = \Phi(YXu) = \Phi(tu) = t\Phi(u) = YX\Phi(u) = Y\Phi(Xu) = Y\Phi(v).$$

It remains to compare the results of applying ΦY and $Y\Phi$ on basis vectors which are not in the image of X . They have the form $[\sigma(\mathbf{m}), e_k]$ where $\mathbf{m} \in B_\omega$ and $z_k \neq x$, i.e. $z_k = y$ or $k = 0$.

$$\begin{aligned}
& \left(Y\Phi([\sigma(\mathbf{m}), e_k]) \right) ([\mathbf{m}, e_l]) = \sigma^{-1} \left(c_{\sigma(\mathbf{m}),k} [\sigma(\mathbf{m}), e_k^\#] (X[\mathbf{m}, e_l]) \right) = \\
& = \begin{cases} \sigma^{-1} \left(c_{\sigma(\mathbf{m}),k} [\sigma(\mathbf{m}), e_k^\#] ([\sigma(\mathbf{m}), e_{l+1}]) \right), & z_{l+1}^\# = x, \\ 0, & \text{otherwise} \end{cases} \\
& = \begin{cases} \sigma^{-1} (c_{\sigma(\mathbf{m}),k}) \delta_{k,l+1} \cdot 1_{\mathbf{m}}, & z_{l+1} = y, \\ 0, & \text{otherwise} \end{cases} \\
& = \begin{cases} \sigma^{-1} (c_{\sigma(\mathbf{m}),k}) c_{\mathbf{m},k-1}^{-1} \left(\Phi([\mathbf{m}, e_{k-1}]) \right) ([\mathbf{m}, e_l]), & z_k = y, \\ 0, & \text{otherwise} \end{cases} \\
& = \left(\Phi(Y[\sigma(\mathbf{m}), e_k]) \right) ([\mathbf{m}, e_l])
\end{aligned}$$

if the coefficients are chosen such that $\sigma^{-1}(c_{\sigma(\mathbf{m}),k})/c_{\mathbf{m},k-1} = 1$ when $\mathbf{m} \in B_\omega$ and $z_k = y$. Choosing the coefficients in this way, which is always possible, Φ becomes an isomorphism of A -modules. \square

Example 5.9. Assume that $\omega \in \Omega$ is real and $p = |\omega| = 7$. Pick $\mathbf{n} \in \omega$. Then $\omega = \{\sigma^j(\mathbf{n}) \mid j = 0, \dots, 6\}$. Suppose that $B_\omega = \{\mathbf{m}_0 := \sigma^2(\mathbf{n}), \mathbf{m}_1 := \sigma^4(\mathbf{n}), \mathbf{m}_2 := \sigma^6(\mathbf{n})\}$, so and $m = |B_\omega| = 3$. The following is a weight diagram for $V(\omega, j, w)$ where $j = 0$ and $w = z_1 z_2 \cdots z_{10}$.



With ω as above, there are three modules of the form $V(\omega, j, \varepsilon)$ corresponding to $j = 0, 1, 2$. For example, $V(\omega, 1, \varepsilon)$ is two-dimensional with basis $\{[\sigma^{-1}(\mathbf{m}_1), e_1], [\mathbf{m}_1, e_1]\}$.

In general, let $j \in \mathbb{Z}_m$ and $V = V(\omega, j, \varepsilon)$. We determine all non-degenerate admissible forms on V . V has a basis

$$\{v_k := [\sigma^{-k}(\mathbf{m}_j), e_j] \mid k = 0, 1, \dots, p_j - 1\},$$

where $p_j > 0$ is minimal such that $\sigma^{p_j}(\mathbf{m}_{j-1}) = \mathbf{m}_j$. Any A -module isomorphism $V \rightarrow V^\#$ has the form $\Phi_\lambda(v_0) = \lambda v_0^\#$ for some $\lambda \in \mathbb{K}_{\mathbf{m}_j}$, where $v_0^\# = [\mathbf{m}_j, e_j^\#]$. The corresponding admissible form satisfies

$$\begin{aligned} \widehat{\Phi}_\lambda(v_n, v_m) &= \widehat{\Phi}_\lambda(Y^n v_0, Y^m v_0) \delta_{n,m} = \sigma^{-n}(\widehat{\Phi}_\lambda(X^n Y^m v_0, v_0)) \delta_{n,m} = \\ &= \sigma^{-n}(\sigma(t) \sigma^2(t) \cdots \sigma^n(t) \lambda) \delta_{n,m} \end{aligned} \quad (5.15)$$

for $n, m = 0, 1, \dots, p_j - 1$. It is clearly non-degenerate iff $\lambda \neq 0$.

Suppose that ω is torsion trivial. Choose $\mathbf{m}(\omega) = \mathbf{m}_j$. Suppose that $\mathbb{K}_\omega \simeq \mathbb{C}$ and that conjugation is usual complex conjugation and assume that $\lambda \in \mathbb{R}$. Let Ψ_λ be the associated symmetric \mathbb{C} -form as described in Proposition 3.4. We have

$$\Psi_\lambda(v_n, v_m) = (\sigma(t) \sigma^2(t) \cdots \sigma^n(t))_{\mathbf{m}(\omega)} \lambda \delta_{n,m}$$

for $n, m = 0, 1, \dots, p_j - 1$. Let us calculate the index (n_+, n_-) , (i.e. n_+ (n_-) is the number of positive (negative) eigenvalues) of the form Ψ_λ . Let $a_0 = \lambda$ and

$a_i = \sigma^i(t) + \mathbf{m}(\omega) \in \mathbb{R}$, $i = 1, \dots, p_j - 1$. Let $0 \leq s_1 < s_2 < \dots < s_r \leq p_j - 1$ be the integers i for which $a_i < 0$ and put $s_i = 0$ for $i \leq 0$ and put $s_i = p_j$ for $i > r$. Then one can check that Ψ_λ has index $(\sum_{i \in \mathbb{Z}} (s_{2i+1} - s_{2i}), \sum_{i \in \mathbb{Z}} (s_{2i} - s_{2i-1}))$. For example, if $p_j = 7$ and $\text{sgn}(\lambda, a_1, a_2, \dots, a_6) = (+, +, -, +, +, -, -)$, then the index of Ψ_λ is $(2 + 1, 3 + 1) = (3, 4)$. All possible indices can occur. This can be seen as in Section 5.1.

5.5 Finite orbit with breaks, second kind

For $r \in R$ and $\mathbf{m} \in \text{Max}(R)$, we put $r_{\mathbf{m}} = r + \mathbf{m} \in R/\mathbf{m}$ for brevity.

Theorem 5.10. *Let $\omega \in \Omega$ be a finite real orbit. Let $V = V(\omega, w, f)$ where $w = z_1 z_2 \dots z_n$ is an m -word, and $f = a_1 + a_2 x + \dots + a_d x^{d-1} + x^d \in \mathbb{K}_\omega[x; \tau^{n/m}]$ is any element with $a_1 \neq 0$. Then $V^\sharp \simeq V(\omega, w^\sharp, g)$ for some $g \in \mathbb{K}_\omega[x; \tau^{n/m}]$.*

Proof. For simplicity, we will assume that $z_1 = x$. The proof of the case $z_1 = y$ is similar.

Step 1. We find the action of X and Y on a dual basis in V^\sharp . Relations (4.3)-(4.4) for the module V can be written

$$X[\mathbf{m}, e_{ks}] = \begin{cases} \sigma(t_{\mathbf{m}}) \cdot [\sigma(\mathbf{m}), e_{ks}], & \mathbf{m} \notin B_\omega, \\ [\sigma(\mathbf{m}), e_{k+1,s}], & \mathbf{m} \in B_\omega, k < n, z_{k+1} = x, \\ 0, & \mathbf{m} \in B_\omega, k < n, z_{k+1} = y, \\ [\sigma(\mathbf{m}), e_{1,s+1}], & \mathbf{m} \in B_\omega, k = n, s < d, \\ -\sum_{i=1}^d \sigma(a_i) \cdot [\sigma(\mathbf{m}), e_{1i}], & \mathbf{m} \in B_\omega, k = n, s = d, \end{cases} \quad (5.16)$$

$$Y[\mathbf{m}, e_{ks}] = \begin{cases} [\sigma^{-1}(\mathbf{m}), e_{ks}], & \sigma^{-1}(\mathbf{m}) \notin B_\omega, \\ [\sigma^{-1}(\mathbf{m}), e_{k-1,s}], & \sigma^{-1}(\mathbf{m}) \in B_\omega, k > 1, z_k = y, \\ 0, & \sigma^{-1}(\mathbf{m}) \in B_\omega, k > 1, z_k = x, \\ 0, & \sigma^{-1}(\mathbf{m}) \in B_\omega, k = 1. \end{cases} \quad (5.17)$$

Let

$$\{[\mathbf{m}, e_{ks}^\sharp] \mid s = 1, \dots, d, k = 1, \dots, n, k \equiv j(\mathbf{m}) \pmod{m}\}$$

be the dual basis in V^\sharp , defined by requiring (recall that $1_{\mathbf{m}}$ denotes $1 + \mathbf{m} \in R/\mathbf{m}$)

$$[\mathbf{m}, e_{ks}^\sharp]([\mathbf{n}, e_{lr}]) = \begin{cases} 1_{\mathbf{m}}, & \text{if } \mathbf{m} = \mathbf{n}, k = l, s = r, \\ 0, & \text{otherwise,} \end{cases} \quad (5.18)$$

and $[\mathbf{m}, e_{ks}^\sharp]$ to be additive and $[\mathbf{m}, e_{ks}^\sharp](rv) = r^* \cdot [\mathbf{m}, e_{ks}^\sharp](v)$ for any $r \in R$, $v \in V$. Then the following relations hold for the action of X and Y on this dual basis:

$$X[\mathbf{m}, e_{ks}^\sharp] = \begin{cases} [\sigma(\mathbf{m}), e_{ks}^\sharp], & \mathbf{m} \notin B_\omega, \\ [\sigma(\mathbf{m}), e_{k+1,s}^\sharp], & \mathbf{m} \in B_\omega, k < n, z_{k+1} = y, \\ 0, & \text{otherwise,} \end{cases} \quad (5.19)$$

$$\begin{aligned}
Y[\mathbf{m}, e_{ks}^\#] &= \\
&= \begin{cases} t_{\sigma^{-1}(\mathbf{m})} \cdot [\sigma^{-1}(\mathbf{m}), e_{ks}^\#], & \sigma^{-1}(\mathbf{m}) \notin B_\omega, \\
[\sigma^{-1}(\mathbf{m}), e_{k-1,s}^\#], & \sigma^{-1}(\mathbf{m}) \in B_\omega, k > 1, z_k = x, \\
0, & \sigma^{-1}(\mathbf{m}) \in B_\omega, k > 1, z_k = y, \\
[\sigma^{-1}(\mathbf{m}), e_{n,s-1}^\#] - \overline{a_s} \cdot [\sigma^{-1}(\mathbf{m}), e_{nd}^\#], & \sigma^{-1}(\mathbf{m}) \in B_\omega, k = 1, s > 1, \\
-\overline{a_1} \cdot [\sigma^{-1}(\mathbf{m}), e_{nd}^\#], & \sigma^{-1}(\mathbf{m}) \in B_\omega, k = 1, s = 1. \end{cases}
\end{aligned} \tag{5.20}$$

Let us prove the first case in (5.20). If $\sigma^{-1}(\mathbf{m}) \notin B_\omega$, then

$$\begin{aligned}
(Y[\mathbf{m}, e_{ks}^\#])([\sigma^{-1}(\mathbf{m}), e_{lr}]) &= \\
&= \sigma^{-1}([\mathbf{m}, e_{ks}^\#](X[\sigma^{-1}(\mathbf{m}), e_{lr}])) \quad \text{by } A\text{-module str. of } V^\#, \\
&= \sigma^{-1}([\mathbf{m}, e_{ks}^\#](\sigma(t) \cdot [\mathbf{m}, e_{lr}])) \quad \text{by (5.16),} \\
&= \sigma^{-1}(\sigma(t)^* \cdot [\mathbf{m}, e_{ks}^\#](\mathbf{m}, e_{lr})) \quad \text{by } R\text{-antilinearity,} \\
&= t \cdot \delta_{kl} \delta_{sr} \cdot \sigma^{-1}(1_{\mathbf{m}}) \quad \text{by (5.18),} \\
&= t \cdot [\sigma^{-1}(\mathbf{m}), e_{ks}^\#](\sigma^{-1}(\mathbf{m}), e_{lr}) \quad \text{by (5.18).}
\end{aligned}$$

Furthermore, if $\mathbf{n} \neq \sigma^{-1}(\mathbf{m})$ then

$$(Y[\mathbf{m}, e_{ks}^\#])([\mathbf{n}, e_{lr}]) = \sigma^{-1}([\mathbf{m}, e_{ks}^\#](X[\mathbf{n}, e_{lr}])) = 0 = t \cdot [\sigma^{-1}(\mathbf{m}), e_{ks}^\#](\mathbf{n}, e_{lr})$$

using that $X[\mathbf{n}, e_{lr}] \in V_{\sigma(\mathbf{n})}$ and (5.18). This proves that $Y[\mathbf{m}, e_{ks}^\#] = t \cdot [\sigma^{-1}(\mathbf{m}), e_{ks}^\#] = t_{\sigma^{-1}(\mathbf{m})} \cdot [\sigma^{-1}(\mathbf{m}), e_{ks}^\#]$ if $\sigma^{-1}(\mathbf{m}) \notin B_\omega$.

For the last two cases in (5.20), let us first note that if $\sigma^{-1}(\mathbf{m}) \in B_\omega$ and $j(\sigma^{-1}(\mathbf{m})) \equiv n \equiv 0 \pmod{m}$ then in fact $\sigma^{-1}(\mathbf{m}) = \mathbf{m}_0$. We have

$$\begin{aligned}
(Y[\sigma(\mathbf{m}_0), e_{1s}^\#])([\mathbf{m}_0, e_{lr}]) &= \\
&= \sigma^{-1}([\sigma(\mathbf{m}_0), e_{1s}^\#](X[\mathbf{m}_0, e_{lr}])) \quad \text{by } A\text{-module str. of } V^\#, \\
&= \sigma^{-1}([\sigma(\mathbf{m}_0), e_{1s}^\#](\sigma(\mathbf{m}_0), e_{1,r+1}) \delta_{ln} \delta_{r < d} - \sigma(a_s) [\sigma(\mathbf{m}_0), e_{1s}] \delta_{ln} \delta_{rd})) \\
&= \delta_{s-1,r} \delta_{s>1} \delta_{ln} 1_{\mathbf{m}_0} - \overline{a_s} \delta_{ln} \delta_{rd} 1_{\mathbf{m}_0} \\
&= ([\mathbf{m}_0, e_{n,s-1}^\#] \delta_{s>1} - \overline{a_s} \cdot [\mathbf{m}_0, e_{nd}^\#])([\mathbf{m}_0, e_{lr}]).
\end{aligned}$$

The other cases in (5.19), (5.20) are easily checked.

Step 2. We construct a basis $[\mathbf{m}, f_{ks}]$ for $V^\#$ such that $[\mathbf{m}, e_{ks}] \mapsto [\mathbf{m}, f_{ks}]$ is an isomorphism from $V(\omega, w^\#, g)$ to $V^\#$ for some g . We have a decomposition

$$(V^\#)_{\mathbf{m}} = \bigoplus_{\substack{1 \leq k \leq n, \\ k \equiv j(\mathbf{m}) \pmod{m}}} (V^\#)^{(k)}_{\mathbf{m}} \quad \text{for any } \mathbf{m} \in \omega, \tag{5.21}$$

$$(V^\sharp)_{\mathbf{m}}^{(k)} = \oplus_{s=1}^d \mathbb{K}_{\mathbf{m}}[\mathbf{m}, e_{ks}^\sharp]. \quad (5.22)$$

Note that, if $k > 1$ and $z_k^\sharp = y$ then $Y : (V^\sharp)_{\mathbf{m}}^{(k)} \rightarrow (V^\sharp)_{\sigma^{-1}(\mathbf{m})}^{(k-1)}$ is bijective, where $\sigma^{-1}(\mathbf{m}) \in B_\omega$ is the unique break such that $j(\mathbf{m}) \equiv k \pmod{m}$. Indeed this is trivial since $Y[\mathbf{m}, e_{ks}^\sharp] = [\sigma^{-1}(\mathbf{m}), e_{k-1,s}^\sharp]$ for $s = 1, \dots, d$ by the second case in (5.20). Also, $Y : (V^\sharp)_{\sigma(\mathbf{m}_0)}^{(1)} \rightarrow (V^\sharp)_{\mathbf{m}_0}^{(n)}$ is bijective by the fourth and fifth case in (5.20), using the assumption that $a_1 \neq 0$.

Put

$$[\sigma(\mathbf{m}_0), f_{11}] = [\sigma(\mathbf{m}_0), e_{11}^\sharp] \quad (5.23)$$

and recursively

$$[\mathbf{m}, f_{ks}] = \begin{cases} \sigma(t)_{\mathbf{m}}^{-1} X[\sigma^{-1}(\mathbf{m}), f_{ks}], & \sigma^{-1}(\mathbf{m}) \notin B_\omega, \\ X[\sigma^{-1}(\mathbf{m}), f_{k-1,s}], & \sigma^{-1}(\mathbf{m}) \in B_\omega, z_k^\sharp = x (\Rightarrow k > 1), \\ (Y|_{(V^\sharp)_{\mathbf{m}}^{(k)}})^{-1} [\sigma^{-1}(\mathbf{m}), f_{k-1,s}], & \sigma^{-1}(\mathbf{m}) \in B_\omega, k > 1, z_k^\sharp = y, \\ (Y|_{(V^\sharp)_{\mathbf{m}}^{(1)}})^{-1} [\sigma^{-1}(\mathbf{m}), f_{n,s-1}], & \sigma^{-1}(\mathbf{m}) \in B_\omega, k = 1. \end{cases} \quad (5.24)$$

Induction shows that each $[\mathbf{m}, f_{ks}]$ is a linear combination of $[\mathbf{m}, e_{kr}^\sharp]$ where $1 \leq r \leq s$ and the coefficient of $[\mathbf{m}, e_{ks}^\sharp]$ is nonzero. Thus $\{[\mathbf{m}, f_{ks}]\}_{s=1}^d$ is a basis for $(V^\sharp)_{\mathbf{m}}^{(k)}$.

We prove that there exists a $g \in \mathbb{K}_\omega[x; \tau^{n/m}]$ such that the R -module isomorphism $\varphi : V(\omega, w^\sharp, g) \rightarrow V^\sharp$ defined by $\varphi([\mathbf{m}, e_{ks}]) = [\mathbf{m}, f_{ks}]$ is an A -module isomorphism. By (4.3),

$$\begin{aligned} \varphi(X[\mathbf{m}, e_{ks}]) &= \begin{cases} \varphi(\sigma(t)_{\sigma(\mathbf{m})} \cdot [\sigma(\mathbf{m}), e_{ks}]), & \mathbf{m} \notin B_\omega, \\ \varphi([\sigma(\mathbf{m}), e_{k+1,s}]), & \mathbf{m} \in B_\omega, k < n, z_{k+1}^\sharp = x, \\ 0, & \text{otherwise (since } z_1^\sharp = y), \end{cases} \\ &= \begin{cases} \sigma(t)_{\sigma(\mathbf{m})} \cdot [\sigma(\mathbf{m}), f_{ks}], & \mathbf{m} \notin B_\omega, \\ [\sigma(\mathbf{m}), f_{k+1,s}], & \mathbf{m} \in B_\omega, k < n, z_{k+1}^\sharp = x, \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \quad (5.25)$$

while $X\varphi([\mathbf{m}, e_{ks}]) = X[\mathbf{m}, f_{ks}]$. By the recursive definition of $[\mathbf{m}, f_{ks}]$, the vector $X[\mathbf{m}, f_{ks}]$ equals the right hand side of (5.25). For example, $[\sigma(\mathbf{m}), f_{ks}] = \sigma(t)_{\sigma(\mathbf{m})}^{-1} \cdot X[\mathbf{m}, f_{ks}]$ if $\mathbf{m} \notin B_\omega$ by the first case in (5.24), which gives $X[\mathbf{m}, f_{ks}] = \sigma(t)_{\sigma(\mathbf{m})} \cdot [\sigma(\mathbf{m}), f_{ks}]$. Similarly, by (4.4) and the construction of the basis $[\mathbf{m}, f_{ks}]$, $\varphi(Y[\mathbf{m}, e_{ks}]) = Y\varphi([\mathbf{m}, e_{ks}])$ when $k > 1$ or $s > 1$ or $\mathbf{m} \neq \sigma(\mathbf{m}_0)$. For the last case, $k = s = 1$ and $\mathbf{m} = \sigma(\mathbf{m}_0)$, we know that $Y : (V^\sharp)_{\sigma(\mathbf{m}_0)}^{(1)} \rightarrow (V^\sharp)_{\mathbf{m}_0}^{(n)}$ is bijective. Thus, since $\{[\mathbf{m}_0, f_{ns}]\}_{s=1}^d$ is a basis for $(V^\sharp)_{\mathbf{m}_0}^{(n)}$,

$$Y\varphi([\sigma(\mathbf{m}_0), e_{11}]) = Y[\sigma(\mathbf{m}_0), f_{11}] = - \sum_{r=1}^d c_r^\circ \cdot [\mathbf{m}_0, f_{nr}]$$

for some constants $c_r \in \mathbb{K}_\omega$, where we denote $c_r^\circ = \tau^{d-r}(c_{d+1-r})$. Choose $g = c_1 + c_2x + \cdots + c_dx^{d-1} + x^d$. Since $z_1^\# = y$, relation (4.4) gives that, in $V(\omega, w^\#, g)$ we have $Y[\sigma(\mathbf{m}_0), e_{11}] = -\sum_{r=1}^d c_r^\circ[\mathbf{m}_0, e_{nr}]$ and thus

$$\varphi(Y[\sigma(\mathbf{m}_0), e_{11}]) = \varphi\left(-\sum_{r=1}^d c_r^\circ[\mathbf{m}_0, e_{nr}]\right) = -\sum_{r=1}^d c_r^\circ[\mathbf{m}_0, f_{nr}].$$

This finishes the proof that $V^\# \simeq V(\omega, w^\#, g)$ for some g . \square

Corollary 5.11. *Let ω be a finite real orbit. Let $V = V(\omega, w, f)$ where $w = z_1z_2\cdots z_n$ is a non-periodic m -word, and $f = a_1 + a_2x + \cdots + a_dx^{d-1} + x^d \neq x^d$ is indecomposable in $\mathbb{K}_\omega[x; \tau^{n/m}]$. If $V \simeq V^\#$ then $w = w_0w_0^\#$, where w_0 is an m -word.*

Proof. Since f is indecomposable and $f \neq x^d$ we have $a_1 \neq 0$. If $V \simeq V^\#$ then by Theorem 5.10, $V \simeq V(\omega, w^\#, g)$ for some $g \in \mathbb{K}_\omega[x; \tau^{n/m}]$. Thus by the classification in Theorem 4.1 we must have $w(lm) = w^\#$ for some integer $l \geq 0$, chosen minimal. Clearly, $lm < n$. Since the operation $\#$ on the monoid \mathbf{D} commutes with the \mathbb{Z} -action, we have

$$w(lm + k) = w(k)^\# \quad \forall k \in \mathbb{Z}. \quad (5.26)$$

We claim that $2lm \leq n$. Otherwise $lm < n < 2lm$ and thus $0 < n - lm < lm$. Also, $w(n - lm) = w(-lm) = w^\#$ since $w = w(-lm + lm) = w(-lm)^\#$ by (5.26) with $k = -lm$. Thus the properties of the number $\frac{n}{m} - l$ contradicts the minimality of l . Therefore $2lm \leq n$ as claimed.

Now let $k = \text{GCD}(2lm, n)$. Trivially $w(n) = w$, and by (5.26), $w(2lm) = w(lm)^\# = w$. Hence $w(k) = w$ also. But $k|n$ and thus $w = (z_1z_2\cdots z_k)^{n/k}$. However w is non-periodic and thus $n = k$, forcing $n = 2lm$ so $w = w_0w_0^\#$ where $w_0 = z_1z_2\cdots z_{lm}$ is an m -word. \square

Theorem 5.12. *Let $\omega \in \Omega$ be a finite real orbit with $m := |B_\omega| > 0$. Let $w_0 \in \mathbf{D} \setminus \{\varepsilon\}$ be an m -word and put $l = |w_0|/m$ and $n = 2|w_0|$. Let $V = V(\omega, w_0w_0^\#, f)$ where $f = \alpha_0 + \alpha_1x + \cdots + \alpha_{d-1}x^{d-1} + \alpha_dx^d \in \mathbb{K}_\omega[x; \tau^{n/m}]$ is any element with $\alpha_0 \neq 0 \neq \alpha_d$. Then $V^\# \simeq V(\omega, w_0w_0^\#, f^\#)$, where*

$$f^\# = \sum_{k=0}^d \{2lk\} \cdot \tau^{(2k+1)l}(\overline{\alpha_{d-k}}) \cdot x^k. \quad (5.27)$$

Here $\{k\}$ is a Pochhammer-type symbol:

$$\{k\} = \{k\}_{q,\tau} = q\tau(q) \cdots \tau^{k-1}(q) \in \mathbb{K}_\omega, \quad k \in \mathbb{Z}_{\geq 0}, \quad (5.28)$$

where $q \in \mathbb{K}_\omega \setminus \{0\}$ is given by

$$q = \sigma^{p_2+p_3+\cdots+p_m}(t_1)\sigma^{p_3+p_4+\cdots+p_m}(t_2) \cdots \sigma^{p_m}(t_{m-1})t_m, \quad (5.29)$$

$$t_i = (\sigma(t)\sigma^2(t) \cdots \sigma^{p_i-1}(t))_{\mathbf{m}_i} \quad \text{for } i = 1, \dots, m, \quad (5.30)$$

where $p_i \in \mathbb{Z}_{>0}$ are minimal such that $\sigma^{p_i}(\mathbf{m}_{i-1}) = \mathbf{m}_i$, $i = 1, \dots, m$.

Combining Corollary 5.11 and Theorem 5.12 we obtain the following.

Theorem 5.13. *Let V be any indecomposable weight A -module of the type $V(\omega, w, f)$ with ω real. Thus $\omega \in \Omega$ is a finite real orbit with $m := |B_\omega| > 0$, $w \in \mathbf{D} \setminus \{\varepsilon\}$ is a non-periodic m -word, and $f = \alpha_0 + \alpha_1 x + \cdots + \alpha_d x^d \in \mathbb{K}_\omega[x; \tau^{n/m}]$, $\alpha_d \neq 0$, is an indecomposable element not equal to x^d . Then V has a non-degenerate admissible form iff $w = w_0 w_0^\sharp$ for some m -word $w_0 \in \mathbf{D} \setminus \{\varepsilon\}$ and f is similar to f^\sharp in $\mathbb{K}_\omega[x; \tau^{n/m}]$, where f^\sharp is given by (5.27).*

Remark 5.14. From Theorem 5.12 follows that $f^{\sharp\sharp}$ is similar to f . This is not apparent from (5.27) but by comparing the coefficients of f and $f^{\sharp\sharp}$ one can verify that

$$f^{\sharp\sharp} = \{(2d+1)l\} \cdot \tau^{\frac{n}{m}(m+1)}(f) \cdot \{l\}^{-1}.$$

Using that $\tau^{n/m}(f)$ is similar to f in $\mathbb{K}_\omega[x; \tau^{n/m}]$ we conclude that indeed $f^{\sharp\sharp} \sim f$.

Proof of Theorem 5.12. Let $z_1 z_2 \cdots z_n = w$. It will also be convenient to define $z_j = z_i$ when $j \equiv i \pmod{n}$. Assume for a moment that we have proved (5.27) for the case $z_1 = x$ and suppose that $z_1 = y$. By the shift isomorphism (4.6), which holds also for decomposable f , we have

$$V \simeq V(\omega, w(-lm), \tau^{-l}(f)) = V(\omega, w_0^\sharp w_0, \tau^{-l}(f)) \quad (5.31)$$

where $\tau^{-l}(f) = \tau^{-l}(\alpha_0) + \tau^{-l}(\alpha_1)x + \cdots + \tau^{-l}(\alpha_d)x^d$. By the assumption we then have

$$V(\omega, w_0^\sharp w_0, \tau^{-l}(f))^{\sharp} \simeq V(\omega, w_0^\sharp w_0, g), \quad (5.32)$$

where

$$g = \sum_{k=0}^d \{2lk\} \cdot \tau^{(2k+1)l} \left(\overline{\tau^{-l}(\alpha_{d-k})} \right) \cdot x^k = \sum_{k=0}^d \tau^{-l} \left(\tau^l(\{2lk\}) \cdot \tau^{(2k+1)l} \left(\overline{\alpha_{d-k}} \right) \right) \cdot x^k.$$

Again by (4.6),

$$V(\omega, w_0^\sharp w_0, g) \simeq V(\omega, w_0 w_0^\sharp, \tau^l(g)). \quad (5.33)$$

From the formula

$$\tau^l(\{2lk\}) = \{l\}^{-1} \cdot \{2lk\} \cdot \tau^{2lk}(\{l\})$$

we see that $\tau^l(g) = \{l\}^{-1} \cdot f^\sharp \cdot \{l\}$ which is similar to f^\sharp . Combining this fact with the isomorphisms (5.31)-(5.33) we deduce that $V^\sharp \simeq V(\omega, w, f^\sharp)$. Therefore the case $z_1 = y$ follows from the case $z_1 = x$.

Thus we assume for the rest of the proof that $z_1 = x$.

Step 1. Put $a_k = \alpha_{k-1}/\alpha_d$ for $k = 1, 2, \dots, d$. Let us replace f by $(1/\alpha_d)f = a_1 + a_2 x + \cdots + a_d x^{d-1} + x^d$. This does not change the

isomorphism class of the module V . As in the proof of Theorem 5.10, we can construct a basis $[\mathbf{m}, f_{ks}]$ for V^\sharp such that

$$\begin{aligned} \varphi : V(\omega, w_0 w_0^\sharp, g) &\rightarrow V^\sharp \\ [\mathbf{m}, e_{ks}] &\mapsto [\mathbf{m}, f_{ks}] \end{aligned} \quad (5.34)$$

is an A -module isomorphism for some g . We use the decomposition (5.21). We put also $(V^\sharp)_{\mathbf{m}}^{(l)} = (V^\sharp)_{\mathbf{m}}^{(k)}$ whenever $l \in \mathbb{Z}, l \equiv k \pmod{n}$. By relation (5.20), which holds in V^\sharp since $z_1 = x$, it follows that if $1 \leq k \leq n$ and $z_k = y$, so that $z_{lm+k} = z_k^\sharp = x$, then

$$Y : (V^\sharp)_{\sigma(\mathbf{m}_{k-1})}^{(lm+k)} \rightarrow (V^\sharp)_{\mathbf{m}_{k-1}}^{(lm+k-1)}$$

is bijective. For the case $k = lm + 1$ it is essential that $a_1 \neq 0$. Put

$$[\sigma(\mathbf{m}_0), f_{11}] = [\sigma(\mathbf{m}_0), e_{lm+1,1}^\sharp] \quad (5.35)$$

and recursively

$$[\mathbf{m}, f_{ks}] = \begin{cases} \sigma(t)_{\mathbf{m}}^{-1} X[\sigma^{-1}(\mathbf{m}), f_{ks}], & \sigma^{-1}(\mathbf{m}) \notin B_\omega, \\ X[\sigma^{-1}(\mathbf{m}), f_{k-1,s}], & \sigma^{-1}(\mathbf{m}) \in B_\omega, k > 1, z_k = x, \\ (Y|_{(V^\sharp)_{\mathbf{m}}^{(k+lm)}})^{-1}[\sigma^{-1}(\mathbf{m}), f_{k-1,s}], & \sigma^{-1}(\mathbf{m}) \in B_\omega, z_k = y, (k > 1), \\ X[\sigma^{-1}(\mathbf{m}), f_{1,s-1}], & \sigma^{-1}(\mathbf{m}) \in B_\omega, k = 1, (z_1 = x). \end{cases} \quad (5.36)$$

By induction, $[\mathbf{m}, f_{ks}] \in (V^\sharp)_{\mathbf{m}}^{(lm+k)}$ for each $\mathbf{m} \in \omega$, $s = 1, \dots, d$, $k = 1, \dots, n$, $k \equiv j(\mathbf{m}) \pmod{m}$.

Step 2. We will now show that the g such that $V(\omega, w_0 w_0^\sharp, g) \simeq V^\sharp$, is similar to f^\sharp , given by (5.27). Define an operator $Z : (V^\sharp)_{\mathbf{m}_0}^{(lm)} \rightarrow (V^\sharp)_{\mathbf{m}_0}^{(lm)}$ by

$$Z = Z_n \cdots Z_2 Z_1, \quad (5.37)$$

where $Z_i : (V^\sharp)_{\mathbf{m}_{i-1}}^{(lm+i-1)} \rightarrow (V^\sharp)_{\mathbf{m}_i}^{(lm+i)}$ are given by

$$Z_i = \begin{cases} (t_i)^{-1} X^{p_i}, & \text{if } z_i = x, \\ (t_i)^{-1} X^{p_i-1} (Y|_{(V^\sharp)_{\sigma(\mathbf{m}_{i-1})}^{(lm+i)}})^{-1}, & \text{if } z_i = y. \end{cases} \quad (5.38)$$

Recall that $\mathbf{m}_0, \mathbf{m}_1, \dots, \mathbf{m}_{m-1}$ are the breaks in ω , ordered such that $\mathbf{m}_{i-1} < \mathbf{m}_i < \mathbf{m}_{i+1}$ for $0 < i < m-1$. See also the weight diagram in Figure 1. For an interpretation of the operator Z , see Remark 5.15. It has the following properties:

$$Z[\mathbf{m}_0, e_{lm,1}^\sharp] = [\mathbf{m}_0, f_{n1}], \quad (5.39)$$

$$[\mathbf{m}_0, f_{ns}] = Z^{s-1}[\mathbf{m}_0, f_{n1}], \quad \text{for } s = 1, 2, \dots, d. \quad (5.40)$$

Let us prove (5.39). We have $Z = Z_n \cdots Z_2 Z_1$. First we prove that

$$Z_1[\mathfrak{m}_0, e_{lm,1}^\#] = [\mathfrak{m}_1, f_{11}]. \quad (5.41)$$

Since $z_1 = x$, and using relation (5.19) and that $z_{lm+1} = z_1^\# = y$, we have

$$Z_1[\mathfrak{m}_0, e_{lm,1}^\#] = (t_1)^{-1} X^{p_1}[\mathfrak{m}_0, e_{lm}^\#] = (t_1)^{-1} X^{p_1-1}[\sigma(\mathfrak{m}_0), e_{lm+1,1}^\#].$$

By definition (5.30) of t_1 and of the vector $[\sigma(\mathfrak{m}_0), f_{11}]$, this is equal to

$$(\sigma(t)\sigma^2(t) \cdots \sigma^{p_1-1}(t))_{\mathfrak{m}_1}^{-1} X^{p_1-1}[\sigma(\mathfrak{m}_0), f_{11}].$$

Using that $\sigma(r)_{\sigma(\mathfrak{m})} X v = X r_{\mathfrak{m}} v$ for any weight vector v of weight \mathfrak{m} and any $r \in R$, where $r_{\mathfrak{m}}$ denotes $r + \mathfrak{m} \in R/\mathfrak{m}$ as usual, the expression can be rearranged into (recall that $\sigma^{p_1}(\mathfrak{m}_0) = \mathfrak{m}_1$)

$$(\sigma(t)_{\sigma^{p_1}(\mathfrak{m}_0)}^{-1} X) (\sigma(t)_{\sigma^{p_1-1}(\mathfrak{m}_0)}^{-1} X) \cdots (\sigma(t)_{\sigma^2(\mathfrak{m}_0)}^{-1} X) [\sigma(\mathfrak{m}_0), f_{11}].$$

By the recursive definition, (5.36), this is equal to $[\sigma^{p_1}(\mathfrak{m}_0), f_{11}] = [\mathfrak{m}_1, f_{11}]$, proving (5.41). Similarly one proves that

$$Z_k[\mathfrak{m}_{k-1}, f_{k-1,1}] = [\mathfrak{m}_k, f_{k1}] \quad \text{for } k = 2, 3, \dots, n.$$

Combining this with (5.41), (5.39) is proved.

In the same way one shows that $[\mathfrak{m}_0, f_{ns}] = Z[\mathfrak{m}_0, f_{n,s-1}]$ for $s = 2, 3, \dots, d$. Then (5.40) follows.

Step 3. We have

$$Z[\mathfrak{m}_0, e_{lm,s}^\#] = \begin{cases} \{2l\}^{-1} \cdot (-\tau^l(\overline{a_{s+1}}/\overline{a_1})[\mathfrak{m}_0, e_{lm,1}^\#] + [\mathfrak{m}_0, e_{lm,s+1}^\#]), & \text{if } s < d, \\ -\{2l\}^{-1} \tau^l(1/\overline{a_1})[\mathfrak{m}_0, e_{lm,1}^\#], & \text{if } s = d. \end{cases} \quad (5.42)$$

To prove this, we first prove that if $1 \leq k \leq lm$, so that $lm + k - 1 < n$, then

$$Z_k[\mathfrak{m}_{k-1}, e_{lm+k-1,s}^\#] = (t_k)^{-1} [\mathfrak{m}_k, e_{lm+k,s}^\#] \quad (5.43)$$

for any $1 \leq s \leq d$. Indeed, if $z_k = x$, then

$$\begin{aligned} Z_k[\mathfrak{m}_{k-1}, e_{lm+k-1,s}^\#] &= \\ &= (t_k)^{-1} X^{p_k}[\mathfrak{m}_{k-1}, e_{lm+k-1,s}^\#] && \text{by definition of } Z_k, \\ &= (t_k)^{-1} X^{p_k-1}[\sigma(\mathfrak{m}_{k-1}), e_{lm+k,s}^\#] && \text{by (5.19), since } z_{lm+k} = z_k^\# = y, \\ &= (t_k)^{-1} [\mathfrak{m}_k, e_{lm+k,s}^\#], && \text{by first case in (5.19).} \end{aligned}$$

We used that $\sigma^{p_k}(\mathfrak{m}_{k-1}) = \mathfrak{m}_k$ in the last step. Similarly, if $z_k = y$, then

$$Y[\sigma(\mathfrak{m}_{k-1}), e_{lm+k,s}^\#] = [\mathfrak{m}_{k-1}, e_{lm+k-1,s}^\#]$$

by (5.20) since $z_{lm+k} = z_k^\# = x$ and $1 < lm + k \leq n$. Therefore

$$(Y|_{(V^\#)^{(lm+k)}_{\sigma(\mathfrak{m}_{k-1})}})^{-1}[\mathfrak{m}_{k-1}, e_{lm+k-1,s}^\#] = [\sigma(\mathfrak{m}_{k-1}), e_{lm+k,s}^\#]$$

and

$$\begin{aligned} Z_k[\mathfrak{m}_{k-1}, e_{lm+k-1,s}^\#] &= (t_k)^{-1} X^{p_k-1} (Y|_{(V^\#)^{(lm+k)}_{\sigma(\mathfrak{m}_{k-1})}})^{-1}[\mathfrak{m}_{k-1}, e_{lm+k-1,s}^\#] = \\ &= (t_k)^{-1} X^{p_k-1} [\sigma(\mathfrak{m}_{k-1}), e_{lm+k,s}^\#] = \\ &= (t_k)^{-1} [\mathfrak{m}_k, e_{lm+k,s}^\#]. \end{aligned}$$

This proves (5.43).

Using (5.43) repeatedly for $k = 1, 2, \dots, lm$ while moving the t_i 's to the left, we have

$$\begin{aligned} Z_m Z_{m-1} \cdots Z_2 Z_1 [\mathfrak{m}_0, e_{lm,s}^\#] &= \\ &= Z_m Z_{m-1} \cdots Z_2 \cdot (t_1)^{-1} [\mathfrak{m}_1, e_{lm+1,s}^\#] = \\ &= \sigma^{p_2+p_3+\cdots+p_m} (t_1)^{-1} Z_m Z_{m-1} \cdots Z_2 [\mathfrak{m}_1, e_{lm+1,s}^\#] = \cdots = \\ &= \sigma^{p_2+p_3+\cdots+p_m} (t_1)^{-1} \sigma^{p_3+p_4+\cdots+p_m} (t_2)^{-1} \cdots \sigma^{p_m} (t_{m-1})^{-1} \cdot (t_m)^{-1} \cdot \\ &\quad \cdot [\mathfrak{m}_m, e_{lm+m,s}^\#] = \\ &= q^{-1} \cdot [\mathfrak{m}_0, e_{(l+1)m,s}^\#]. \end{aligned}$$

Here we use that, from the definition of Z_k , $Z_k \lambda v = \sigma^{p_k}(\lambda) Z_k v$ for $\lambda \in R/\mathfrak{m}$ and v a weight vector of weight \mathfrak{m} , and σ denotes the map $R/\mathfrak{m} \rightarrow R/\sigma(\mathfrak{m})$ induced by σ . In particular, $Z_m Z_{m-1} \cdots Z_1 \lambda v = \tau(\lambda) Z_m Z_{m-1} \cdots Z_1 v$ since $\tau = \sigma^p$ and $p = p_1 + p_2 + \cdots + p_m$. Therefore, using (5.43) as in the above calculation we get

$$\begin{aligned} Z_{lm} Z_{lm-1} \cdots Z_1 [\mathfrak{m}_0, e_{lm,s}^\#] &= Z_{lm} Z_{lm-1} \cdots Z_{m+1} \cdot q^{-1} [\mathfrak{m}_0, e_{(l+1)m,s}^\#] = \\ &= \tau^{l-1} (q^{-1}) Z_{lm} Z_{lm-1} \cdots Z_{m+1} [\mathfrak{m}_0, e_{(l+1)m,s}^\#] = \\ &\quad \cdots \\ &= \tau^{l-1} (q^{-1}) \tau^{l-2} (q^{-1}) \cdots \tau (q^{-1}) q^{-1} \cdot [\mathfrak{m}_0, e_{2lm,s}^\#] = \\ &= \{l\}^{-1} \cdot [\mathfrak{m}_0, e_{n,s}^\#]. \end{aligned} \tag{5.44}$$

It remains to calculate $Z_{2lm} Z_{2lm-1} \cdots Z_{lm+1} [\mathfrak{m}_0, e_{n,s}^\#]$. To calculate $Z_{lm+1} [\mathfrak{m}_0, e_{n,s}^\#]$ we need to find, by definition of Z_{lm+1} ,

$$(Y|_{(V^\#)^{(1)}_{\sigma(\mathfrak{m}_0)}})^{-1} [\mathfrak{m}_0, e_{n,s}^\#]$$

because $z_{lm+1} = z_1^\# = y$. By (5.20),

$$Y[\sigma(\mathfrak{m}_0), e_{1,s+1}^\#] = [\mathfrak{m}_0, e_{n,s}^\#] - \overline{a_{s+1}} \cdot [\mathfrak{m}_0, e_{n,d}^\#], \quad \text{if } s < d, \tag{5.45}$$

$$Y[\sigma(\mathfrak{m}_0), e_{1,1}^\#] = -\overline{a_1} \cdot [\mathfrak{m}_0, e_{n,d}^\#]. \tag{5.46}$$

Therefore

$$\begin{aligned} & (Y|_{(V^\#)^{(1)}_{\sigma(\mathbf{m}_0)}})^{-1}[\mathbf{m}_0, e_{n,s}^\#] = \\ & = \begin{cases} [\sigma(\mathbf{m}_0), e_{1,s+1}^\#] - \sigma(\overline{a_{s+1}}/\overline{a_1}) \cdot [\sigma(\mathbf{m}_0), e_{1,1}^\#], & s < d, \\ -\sigma(1/\overline{a_1}) \cdot [\sigma(\mathbf{m}_0), e_{1,1}^\#], & s = d. \end{cases} \end{aligned} \quad (5.47)$$

Applying $(t_1)^{-1}X^{p_1-1}$ to both sides of (5.47) we deduce that

$$Z_{lm+1}[\mathbf{m}_0, e_{n,s}^\#] = (t_1)^{-1} \cdot \begin{cases} [\mathbf{m}_1, e_{1,s+1}^\#] - \sigma^{p_1}(\overline{a_{s+1}}/\overline{a_1}) \cdot [\mathbf{m}_1, e_{1,1}^\#], & s < d, \\ -\sigma^{p_1}(1/\overline{a_1}) \cdot [\mathbf{m}_1, e_{1,1}^\#], & s = d. \end{cases} \quad (5.48)$$

Similar to relation (5.43) we have the formula

$$Z_{lm+k}[\mathbf{m}_{k-1}, e_{k-1,s}^\#] = (t_k)^{-1}[\mathbf{m}_k, e_{k,s}^\#] \quad \text{for } 1 < k \leq lm \text{ and } 1 \leq s \leq d, \quad (5.49)$$

which can be proved using (5.19), (5.20). Note that $t_{lm+k} = t_k$ by the notational assumptions on \mathbf{m}_k and t_k . Using (5.49) repeatedly we get

$$\begin{aligned} & Z_{(l+1)m}Z_{(l+1)m-1} \cdots Z_{lm+1}[\mathbf{m}_0, e_{n,s}^\#] = \\ & = q^{-1} \cdot \begin{cases} [\mathbf{m}_0, e_{m,s+1}^\#] - \tau(\overline{a_{s+1}}/\overline{a_1}) \cdot [\mathbf{m}_0, e_{m,1}^\#], & s < d, \\ -\tau(1/\overline{a_1}) \cdot [\mathbf{m}_0, e_{m,1}^\#], & s = d. \end{cases} \end{aligned} \quad (5.50)$$

Repeating we get

$$\begin{aligned} & Z_{2lm}Z_{2lm-1} \cdots Z_{lm+1}[\mathbf{m}_0, e_{n,s}^\#] = \\ & = \{l\}^{-1} \cdot \begin{cases} [\mathbf{m}_0, e_{lm,s+1}^\#] - \tau^l(\overline{a_{s+1}}/\overline{a_1}) \cdot [\mathbf{m}_0, e_{lm,1}^\#], & s < d, \\ -\tau^l(1/\overline{a_1}) \cdot [\mathbf{m}_0, e_{lm,1}^\#], & s = d. \end{cases} \end{aligned} \quad (5.51)$$

Thus, combining (5.44) and (5.51) we obtain (5.42) as desired.

Step 4. Set $b_s = -\overline{a_s}/\overline{a_1}$ for $2 \leq s \leq d$ and $b_1 = -1/\overline{a_1}$. We claim that for $1 \leq s < d$, there are constants $C_{s1}, C_{s2}, \dots, C_{ss} \in \mathbb{K}_\omega$ such that

$$\begin{aligned} [\mathbf{m}_0, f_{ns}] &= C_{s1}\tau^{3l}(b_s)[\mathbf{m}_0, f_{n1}] + \cdots + C_{s,s-1}\tau^{l+2l(s-1)}(b_2)[\mathbf{m}_0, f_{n,s-1}] + \\ &+ C_{s,s}(\tau^l(b_{s+1})[\mathbf{m}_0, e_{lm,1}^\#] + [\mathbf{m}_0, e_{lm,s+1}^\#]) \end{aligned} \quad (5.52)$$

We prove this by induction on s . If $s = 1$ we can take

$$C_{11} = \{2l\}^{-1} \quad (5.53)$$

by (5.39) and (5.42). Assume (5.52) holds for some $s < d - 1$. Then, using (5.40) and that $Z\lambda = \tau^{2l}(\lambda)Z$ for any $\lambda \in \mathbb{K}_{\mathbf{m}_0}$, we have

$$\begin{aligned} [\mathbf{m}_0, f_{n,s+1}] &= Z[\mathbf{m}_0, f_{ns}] = \\ &= \tau^{2l}(C_{s1})\tau^{5l}(b_s)Z[\mathbf{m}_0, f_{n1}] + \cdots + \tau^{2l}(C_{s,s-1})\tau^{l+2ls}(b_2)Z[\mathbf{m}_0, f_{n,s-1}] + \\ &+ \tau^{2l}(C_{s,s})(\tau^{3l}(b_{s+1})Z[\mathbf{m}_0, e_{lm,1}^\#] + Z[\mathbf{m}_0, e_{lm,s+1}^\#]) \end{aligned}$$

By (5.39),(5.40) and (5.42) this equals

$$\begin{aligned} & \tau^{2l}(C_{s,s})\tau^{3l}(b_{s+1})[\mathbf{m}_0, f_{n1}] + \\ & + \tau^{2l}(C_{s1})\tau^{5l}(b_s)[\mathbf{m}_0, f_{n2}] + \cdots + \tau^{2l}(C_{s,s-1})\tau^{l+2ls}(b_2)[\mathbf{m}_0, f_{n,s}] + \\ & + \tau^{2l}(C_{s,s})\{2l\}^{-1} \cdot (\tau^l(b_{s+2})[\mathbf{m}_0, e_{lm,1}^\#] + [\mathbf{m}_0, e_{lm,s+2}^\#]). \end{aligned}$$

Thus we seek the solution to the following system of equations

$$C_{s+1,1} = \tau^{2l}(C_{s,s}), \quad (5.54)$$

$$C_{s+1,r} = \tau^{2l}(C_{s,r-1}), \quad 2 \leq r \leq s, \quad (5.55)$$

$$C_{s+1,s+1} = \tau^{2l}(C_{s,s})\{2l\}^{-1}. \quad (5.56)$$

From (5.56),(5.53) we deduce

$$C_{s,s} = \{2ls\}^{-1} \quad 1 \leq s < d. \quad (5.57)$$

Repeated use of (5.55) gives For $1 \leq r < s < d$ we have

$$\begin{aligned} C_{s,r} &= \tau^{2l}(C_{s-1,r-1}) = \cdots = \tau^{2l(r-1)}(C_{s-r+1,1}) && \text{by (5.55)} \\ &= \tau^{2lr}(C_{s-r,s-r}) && \text{by (5.54)} \\ &= \{2lr\}\{2ls\}^{-1} && \text{by (5.57)}. \end{aligned}$$

Substituting this and (5.57) into (5.52) we obtain that, for $1 \leq s < d$,

$$\begin{aligned} [\mathbf{m}_0, f_{ns}] &= \{2l\}\{2ls\}^{-1} \cdot \tau^{3l}(b_s) \cdot [\mathbf{m}_0, f_{n1}] + \\ &+ \{4l\}\{2ls\}^{-1} \cdot \tau^{5l}(b_{s-1}) \cdot [\mathbf{m}_0, f_{n2}] + \\ &\cdots \\ &+ \{2l(s-1)\}\{2ls\}^{-1} \cdot \tau^{l+2l(s-1)}(b_2) \cdot [\mathbf{m}_0, f_{n,s-1}] + \\ &+ \{2ls\}^{-1}(\tau^l(b_{s+1})[\mathbf{m}_0, e_{lm,1}^\#] + [\mathbf{m}_0, e_{lm,s+1}^\#]) \end{aligned} \quad (5.58)$$

In particular, taking $s = d - 1$ and applying Z we have

$$\begin{aligned} [\mathbf{m}_0, f_{nd}] &= Z[\mathbf{m}_0, f_{n,d-1}] = \\ &= \{4l\}\{2ld\}^{-1} \cdot \tau^{5l}(b_{d-1}) \cdot [\mathbf{m}_0, f_{n2}] + \\ &+ \{6l\}\{2ld\}^{-1} \cdot \tau^{7l}(b_{d-2}) \cdot [\mathbf{m}_0, f_{n3}] + \\ &\cdots \\ &+ \{2l(d-1)\}\{2ld\}^{-1} \cdot \tau^{l+2l(d-1)}(b_2) \cdot [\mathbf{m}_0, f_{n,d-1}] + \\ &+ \{2l\}\{2ld\}^{-1} \cdot (\tau^{3l}(b_d)[\mathbf{m}_0, f_{n1}] + \{2l\}^{-1}\tau^l(b_1)[\mathbf{m}_0, e_{lm,1}^\#]) \end{aligned}$$

where we applied (5.42) in the last term. Hence, using that

$$X[\mathbf{m}_0, e_{lm,1}^\#] = [\sigma(\mathbf{m}_0), f_{11}] = [\sigma(\mathbf{m}_0), e_{lm+1,1}^\#]$$

by (5.19) and that $z_{lm+1} = z_1^\# = y$, together with the relation (recall φ from (5.34))

$$\begin{aligned} X[\mathfrak{m}_0, f_{ns}] &= X\varphi([\mathfrak{m}_0, e_{ns}]) = \varphi(X[\mathfrak{m}_0, e_{ns}]) = \\ &= \varphi([\sigma(\mathfrak{m}_0), e_{1,s+1}]) = [\sigma(\mathfrak{m}_0), f_{1,s+1}] \end{aligned}$$

holding for $s < d$, we obtain that

$$\begin{aligned} X[\mathfrak{m}_0, f_{nd}] &= \sigma(\{2ld\}^{-1}\tau^l(b_1)) \cdot [\sigma(\mathfrak{m}_0), f_{11}] + \\ &+ \sigma(\{2l\}\{2ld\}^{-1}\tau^{3l}(b_d)) \cdot [\sigma(\mathfrak{m}_0), f_{12}] + \\ &+ \sigma(\{4l\}\{2ld\}^{-1}\tau^{5l}(b_{d-1})) \cdot [\sigma(\mathfrak{m}_0), f_{13}] + \\ &\dots \\ &+ \sigma(\{2l(d-1)\}\{2ld\}^{-1}\tau^{l+2l(d-1)}(b_2)) \cdot [\sigma(\mathfrak{m}_0), f_{1d}]. \end{aligned}$$

Resubstituting $b_1 = -1/\overline{a_1} = -\overline{a_d}/\overline{a_0}$ and $b_s = -\overline{a_s}/\overline{a_1} = -\overline{a_{s-1}}/\overline{a_0}$ (for $s > 1$), we conclude that, in view of the final case in relation (5.16), that the map $V(\omega, w_0 w_0^\#, g) \rightarrow V^\#, [\mathfrak{m}, e_{ks}] \mapsto [\mathfrak{m}, f_{ks}]$ will be an A -module isomorphism if g is given by

$$\begin{aligned} \{2ld\} \cdot g &= \tau^l(\overline{a_d}/\overline{a_0}) + \\ &+ \{2l\} \cdot \tau^{3l}(\overline{a_{d-1}}/\overline{a_0}) \cdot x + \\ &+ \{4l\} \cdot \tau^{5l}(\overline{a_{d-2}}/\overline{a_0}) \cdot x^2 + \\ &\dots \\ &+ \{2l(d-1)\} \cdot \tau^{l+2l(d-1)}(\overline{a_1}/\overline{a_0}) \cdot x^{d-1} + \\ &+ \{2ld\} \cdot x^d. \end{aligned}$$

Thus $\{2ld\} \cdot g \cdot \tau^l(\overline{a_0}) = f^\#$ so g is similar to $f^\#$. This finishes the proof that $V^\# \simeq V(\omega, w_0 w_0^\#, f^\#)$. \square

Remark 5.15. The indecomposable weight module $V = V(\omega, w, f)$, $w = z_1 \cdots z_n$, has the the following characterizing properties:

- 1) the operator $Z = Z(w) : V_{\mathfrak{m}_0} \rightarrow V_{\mathfrak{m}_0}$ given by $Z = Z_n \cdots Z_2 Z_1$ where

$$Z_i = \begin{cases} (t_i)^{-1} X^{p_i}, & z_i = x, \\ (t_i)^{-1} X^{p_i-1} Y^{-1}, & z_i = y, \end{cases}$$

is well-defined and single-valued (since w is non-periodic), and

- 2) giving $V_{\mathfrak{m}_0}$ the structure of a module over $\mathbb{K}_\omega[x; \tau^{n/m}]$ by

$$x.v = Zv, \quad v \in V_{\mathfrak{m}_0},$$

there exists a nonzero vector in $V_{\mathfrak{m}_0}$ which is annihilated by f .

What we prove in Theorem 5.10 is that $Z(w^\#)$ is well-defined on the \mathfrak{m}_0 -weight space of $V(\omega, w, f)^\#$, while in Theorem 5.12 we prove that when $V = V(\omega, w_0 w_0^\#, f)$, the space $(V^\#)_{\mathfrak{m}_0}$ contains a nonzero vector annihilated by a skew polynomial similar to $f^\#$. Therefore $V^\# \simeq V(\omega, w_0 w_0^\#, f^\#)$.

6 Examples

6.1 Noncommutative type-A Kleinian singularities

Let $R = \mathbb{C}[H]$ and $\sigma \in \text{Aut}_{\mathbb{C}}(H)$ be given by $\sigma(H) = H - 1$ and $t \in R$ be arbitrary. The generalized Weyl algebra $A = R(\sigma, t)$ was studied in [Bav] and [Hod]. For example, all simple modules (not only weight modules) were classified in [Bav]. Let $*$ be the \mathbb{R} -algebra automorphism of R given by $i^* = -i$, $H^* = H$. Suppose that $t^* = t$ i.e. that $t = f(H)$, where the polynomial f has real coefficients. Since any orbit is infinite, Theorem 5.2 and Theorem 5.3 implies that an indecomposable weight module with real support has a non-degenerate admissible form iff it is simple.

6.2 The enveloping algebra of \mathfrak{sl}_2

Let $R = \mathbb{C}[h, t]$ and let $\sigma \in \text{Aut}_{\mathbb{C}}(R)$ be given by $\sigma(h) = h - 2$, $\sigma(t) = t + h$. Then $A = R(\sigma, t) \simeq U(\mathfrak{sl}_2)$. Define $*$ in $\text{Aut}_{\mathbb{R}}(R)$ by $h^* = h$, $t^* = t$, $i^* = -i$. Here, as in the previous example, all orbits are infinite so indecomposable weight modules with real support are non-degenerately unitarizable iff they are simple.

By induction one checks that $\sigma^n(t) = -n^2 + (h+1)n + t$, $\forall n \in \mathbb{Z}$. Thus, for any $\mu, \alpha \in \mathbb{R}$,

$$\lim_{n \rightarrow \pm\infty} \{\sigma^n(t) \bmod (h - \mu, t - \alpha)\} = \lim_{n \rightarrow \pm\infty} -n^2 + (\mu + 1)n + \alpha = -\infty.$$

In view of formulas (5.1), (5.2), (5.3), this shows that any non-degenerate symmetric admissible form on an infinite-dimensional simple weight module with real support is necessarily indefinite.

On the other hand, on a finite-dimensional simple weight module $V(N)$ (with highest weight $N \in \mathbb{Z}_{\geq 0}$ and of dimension $N + 1$), the form Ψ_λ given by (5.2) with $\lambda > 0$ is positive definite because

$$\sigma^n(t) \bmod (t, h - N) = n(N - n + 1) > 0$$

for $n = 1, 2, \dots, N$ so that $\Psi_\lambda(Y^n e_0, Y^n e_0) > 0$ for $n = 0, 1, \dots, N$.

6.3 The quantum enveloping algebra of \mathfrak{sl}_2

Let $R = \mathbb{C}[K, K^{-1}, t]$ and $q \in \mathbb{C} \setminus \{-1, 0, 1\}$. Define $\sigma \in \text{Aut}_{\mathbb{C}}(R)$ by $\sigma(K) = q^{-2}K$, $\sigma(t) = t + \frac{K - K^{-1}}{q - q^{-1}}$. Then $R(\sigma, t) \simeq U_q(\mathfrak{sl}_2)$. We assume here that q^2 is a root of unity of order $p > 1$. Let $*$ in $\text{Aut}_{\mathbb{R}}(R)$ be given by $K^* = K^{-1}$, $i^* = -i$, $t^* = t$. One verifies that σ commutes with $*$ and that σ has order p . All orbits have p elements and are torsion trivial. Let $\omega \in \Omega$ and $\mathfrak{m} = (K - \mu, t - \alpha) \in \omega$. Then ω is real iff $\mathfrak{m}^* = \mathfrak{m}$ which holds iff $|\mu| = 1$ and $\alpha \in \mathbb{R}$. Assume ω is real and put $\mathfrak{m}(\omega) = \mathfrak{m}$. We identify $\mathbb{K}_\omega = R/\mathfrak{m}$ with \mathbb{C} . The real number

$$\xi = (\sigma(t)\sigma^2(t) \cdots \sigma^p(t))_{\mathfrak{m}} = \prod_{k=0}^{p-1} \left(\alpha + \sum_{i=0}^k \frac{q^{-2i}\mu - q^{2i}\mu^{-1}}{q - q^{-1}} \right) \quad (6.1)$$

is nonzero iff there are no breaks in ω .

Assume that $\xi \neq 0$ and consider the modules $V(\omega, f)$. Since $\sigma^p = \text{Id}$, the skew Laurent polynomial ring $\mathbb{K}_\omega[x, x^{-1}; \tau]$, to which f belongs, is just the ordinary commutative Laurent polynomial ring $P = \mathbb{C}[x, x^{-1}]$. Similarity in P just means equality up to multiplication by nonzero homogenous term. Any indecomposable element in P is similar to $f = (x - a)^d$ for some $a \in \mathbb{C} \setminus \{0\}$, $d \geq 1$. By Theorem 5.6, $V(\omega, f)^\# \simeq V(\omega, f^\#)$ where $f^\# = (\xi x)^d((\xi x)^{-1} - \bar{a})^d = (1 - \bar{a}\xi x)^d \sim (x - (\bar{a}\xi)^{-1})^d$. Thus we conclude that $V(\omega, f)$, where ω is a real orbit without breaks containing $(K - \mu, t - \alpha)$ and $f = (x - a)^d$, has a non-degenerate admissible form iff $a = (\bar{a}\xi)^{-1}$, that is, iff $|a|^2 = \xi^{-1}$, where ξ is given by (6.1). It would be interesting to determine the values of α and μ for which ξ is positive so that $|a|^2 = \xi^{-1}$ can hold. We only note here that for any fixed μ , the quantity ξ is a polynomial of degree p in α with positive leading coefficient and thus $\xi > 0$ if α is sufficiently big.

Assume now that $\xi = 0$. Then ω has breaks and we can assume $\alpha = 0$. Recall that the break $\mathbf{m}_0 = \mathbf{m}(\omega) = \mathbf{m}$. For $k \geq 0$ we have

$$\sigma^{k+1}(t) = t + \sum_{i=0}^k \frac{q^{-2i}K - q^{2i}K^{-1}}{q - q^{-1}}.$$

Thus the reduction modulo \mathbf{m}_0 is

$$(\sigma^{k+1}(t))_{\mathbf{m}_0} = \sum_{i=0}^k \frac{q^{-2i}\mu - q^{2i}\mu^{-1}}{q - q^{-1}} = \frac{(1 - q^{2(k+1)})(1 - \mu^2 q^{-2k})}{\mu q(q - q^{-1})^2} \quad (6.2)$$

This shows that, for $0 \leq k \leq p - 2$,

$$\sigma^{-(k+1)}(\mathbf{m}_0) \in B_\omega \iff \mu^2 = q^{2k}. \quad (6.3)$$

By (6.3) we have

$$B_\omega = \begin{cases} \{\mathbf{m}_0, \mathbf{m}_1 = \sigma^{-(k+1)}(\mathbf{m}_0)\}, & \text{if } \mu^2 = q^{2k} \text{ where } 0 \leq k \leq p - 2, \\ \{\mathbf{m}_0\}, & \text{if } \mu \notin \{\pm 1, \pm q, \dots, \pm q^{p-2}\}, \end{cases}$$

Call μ *generic* if $\mu \notin \{\pm 1, \pm q, \dots, \pm q^{p-2}\}$ and *specific* otherwise. If μ is specific, we let r ($0 \leq r \leq p - 2$) denote the unique integer such that $\mu^2 = q^{2r}$. Let $m = |B_\omega|$. By (6.3), $m = 1$ if μ is generic and $m = 2$ if μ is specific. Recall the definition of p_i from Section 4.1. For specific μ we have $p_1 = p - (r + 1)$ and $p_2 = r + 1$.

By Theorem 5.8, a module of the form $V(\omega, j, w)$ has a non-degenerate admissible form iff it is simple, which holds iff $w = \varepsilon$, the empty word. If μ is generic then there is only one such module, $V(\omega, 0, \varepsilon)$. If μ is specific then there are two such modules, $V(\omega, 0, \varepsilon)$ and $V(\omega, 1, \varepsilon)$.

If $V = V(\omega, w = z_1 \cdots z_n, f = (x - a)^d)$, then by Theorem 5.13, V has a non-degenerate admissible form iff $w = w_0 w_0^\#$ where w_0 is a non-empty m -word

(so for generic μ the word w_0 is arbitrary, while for specific μ , it has to be of even length) and f is similar to f^\sharp in $\mathbb{C}[x]$. Let $(a; s)_i$ denote the shifted factorial

$$(a; s)_i = (1 - a)(1 - as) \cdots (1 - as^{i-1})$$

and for $j < i$ let $(a; s)_i^{(j)}$ denote $(a; s)_i$ but with the factor $(1 - as^j)$ omitted. By (5.27) the polynomial f^\sharp is given by

$$f^\sharp = \sum_{k=0}^d Q^{nk} \overline{\alpha_{d-k}} \cdot x^k = (Q^n x)^d \cdot \overline{f((Q^n x)^{-1})} = (1 - Q^n \bar{a} x)^d \sim (x - (Q^n \bar{a})^{-1})^d,$$

where Q is the nonzero real number given by

$$Q = t_1 = \frac{(q^2; q^2)_{p-1} \cdot (\mu^2; q^{-2})_{p-1}}{(\mu q(q - q^{-1})^2)^{p-1}}, \quad \text{if } \mu \text{ is generic,} \quad (6.4)$$

and

$$Q = \sigma^{p^2}(t_1)t_2 = \frac{(q^2; q^2)_{p-1}^{(r)} \cdot (\mu^2; q^{-2})_{p-1}^{(r)}}{(\mu q(q - q^{-1})^2)^{p-2}}, \quad \text{if } \mu \text{ is specific, } \mu^2 = q^{2r}. \quad (6.5)$$

We conclude that $V = V(\omega, w = z_1 \cdots z_n, f = (x - a)^d)$, (ω a real orbit containing a break $\mathfrak{m} = (t, K - \mu)$) has a non-degenerate admissible form iff $w = w_0 w_0^\sharp$, where $w_0 \in \mathbf{D} \setminus \{\varepsilon\}$ has even length if μ is specific, and $|a|^2 = Q^{-n}$. Since n is even, solutions $a \in \mathbb{C}$ to this equation always exist.

Irreducible representations of $U_q(\mathfrak{sl}_2)$ which are unitarizable with respect to a positive definite form were described in [V]. This corresponds to the case when all the factors in (6.1) are nonnegative.

6.4 When R is a field

We note that in the special case when $R = \mathbb{K}$ is a field, there is only one orbit ω_0 consisting of the zero ideal alone. The orbit ω_0 is real, and contains a break iff $t = 0$. Furthermore, ω_0 is torsion trivial iff σ is trivial. An indecomposable weight module over $A = R(\sigma, t)$ is then of the form $V(\omega, f)$ if $t \neq 0$, where $f \in \mathbb{K}[x, x^{-1}; \sigma]$ and $V(\omega, j, w)$ or $V(\omega, w, f)$ if $t = 0$, where $f \in \mathbb{K}[x; \sigma^n]$ ($n = |w|$). This shows that any skew polynomial ring can occur.

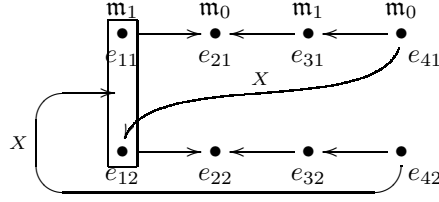
6.5 An example of a module of the second kind

Let $R = \mathbb{C}[u, t]$, $\sigma \in \text{Aut}_{\mathbb{C}}(R)$ defined by $\sigma(u) = 1 - u, \sigma(t) = t$. Then the orbits have the form $\omega_{\mu, \alpha} = \{(u - \mu, t - \alpha), (u - (1 - \mu), t - \alpha)\}$, where $\mu, \alpha \in \mathbb{C}$. All orbits are torsion trivial and have two elements, except for $\omega_{1/2, \alpha}$ which has only one element. The orbit $\omega_{\mu, \alpha}$ contains no breaks if $\alpha \neq 0$, and all elements of $\omega_{\mu, 0}$ are breaks. Define $* \in \text{Aut}_{\mathbb{R}}(R)$ by $u^* = u, t^* = t, i^* = -i$. Then $\omega_{\mu, \alpha}$ is real iff $\mu, \alpha \in \mathbb{R}$.

Let $\omega = \omega_{0,0}$. Let $\mathfrak{m}(\omega) = \mathfrak{m}_0 = (u, t)$ and $\sigma(\mathfrak{m}_0) = \mathfrak{m}_1 = (u-1, t)$. Then $B_\omega = \omega$, $p = |\omega| = 2$, $m = |B_\omega| = 2$. We identify $\mathbb{K}_\omega = R/\mathfrak{m}(\omega)$ with \mathbb{C} . The map τ is the identity since ω is torsion trivial. Let $f = a_1 + a_2x + x^2 \in \mathbb{C}[x]$, $a_1 \neq 0$, let $w = xxyy$ and let $V = V(\omega, w, f)$. The weight module V is decomposable iff f has distinct roots.

Since $\sigma(\mathfrak{m}_0) = \mathfrak{m}_1$ and $\sigma(\mathfrak{m}_1) = \mathfrak{m}_0$, the integers p_1 and p_2 (defined in Section 4.1) both equal one. Thus, recalling definitions (5.29), (5.30) of q , t_1, t_2 , we have $t_1 = t_2 = 1$ and $q = 1$. By Theorem 5.12, $V^\# \simeq V(\omega, w, f^\#)$ where $f^\# = 1 + \overline{a_2}x + \overline{a_1}x^2 \sim 1/\overline{a_1} + \overline{a_2}/\overline{a_1} \cdot x + x^2$. Thus $V \simeq V^\#$ iff $a_1 = 1/\overline{a_1}$, $a_2 = \overline{a_2}/\overline{a_1}$.

The module V has the following structure. We have $V = V_{\mathfrak{m}_0} \oplus V_{\mathfrak{m}_1}$. Since $j(\mathfrak{m}_0) = 0$ and $j(\mathfrak{m}_1) = 1$, $V_{\mathfrak{m}_0}$ has a basis $\{e_{21}, e_{22}, e_{41}, e_{42}\}$ and $V_{\mathfrak{m}_1}$ has a basis $\{e_{11}, e_{12}, e_{31}, e_{32}\}$.



The module structure on V is given by the following, where $s = 1, 2$:

$$\begin{cases} X e_{1s} = e_{2s}, \\ X e_{2s} = X e_{3s} = 0, \\ X e_{41} = e_{12}, \\ X e_{42} = -a_1 e_{11} - a_2 e_{12}, \end{cases} \quad \begin{cases} Y e_{1s} = 0, \\ Y e_{2s} = 0, \\ Y e_{3s} = e_{2s}, \\ Y e_{4s} = e_{3s}. \end{cases}$$

Let us show explicitly that $V^\# \simeq V(\omega, w, f^\#)$. Let $\{e_{ks}^\# : 1 \leq k \leq 4, s = 1, 2\}$ be the dual basis in $V^\#$, i.e. $e_{ks}^\#(e_{ij}) = \delta_{ki}\delta_{sj}$. Then $\{e_{2s}^\#, e_{4s}^\# : s = 1, 2\}$ is a basis for $(V^\#)_{\mathfrak{m}_0}$ and $\{e_{1s}^\#, e_{3s}^\# : s = 1, 2\}$ is a basis for $(V^\#)_{\mathfrak{m}_1}$. For $s = 1, 2$ we have

$$\begin{cases} X e_{1s}^\# = 0, \\ X e_{2s}^\# = e_{3s}^\#, \\ X e_{3s}^\# = e_{4s}^\#, \\ X e_{4s}^\# = 0, \end{cases} \quad \begin{cases} Y e_{11}^\# = -\overline{a_1} e_{42}^\#, \\ Y e_{12}^\# = e_{41}^\# - \overline{a_2} e_{42}^\#, \\ Y e_{2s}^\# = e_{1s}^\#, \\ Y e_{3s}^\# = Y e_{4s}^\# = 0. \end{cases}$$

Set $b_1 = -1/\overline{a_1}$ and $b_2 = -\overline{a_2}/\overline{a_1}$ and

$$\begin{cases} f_{11} = e_{31}^\#, \\ f_{21} = e_{41}^\#, \\ f_{31} = b_2 e_{11}^\# + e_{12}^\#, \\ f_{41} = b_2 e_{21}^\# + e_{22}^\#, \end{cases} \quad \begin{cases} f_{12} = b_2 e_{31}^\# + e_{32}^\#, \\ f_{22} = b_2 e_{41}^\# + e_{42}^\#, \\ f_{32} = (b_1 + b_2^2) e_{11}^\# + b_2 e_{12}^\#, \\ f_{42} = (b_1 + b_2^2) e_{21}^\# + b_2 e_{22}^\#. \end{cases} \quad (6.6)$$

We have $X f_{42} = b_1 f_{11} + b_2 f_{12}$. Set $g(x) = -b_1 - b_2 x + x^2$. Then one verifies that $V^\# \simeq V(\omega, w, g)$ via the map $f_{ks} \mapsto e_{ks}^\#$. Since $g \sim f^\#$ we deduce that

$V^\sharp \simeq V(\omega, w, f^\sharp)$. Thus, since polynomials in $\mathbb{C}[x]$ are similar iff they differ by a multiplicative scalar, $V \simeq V^\sharp$ iff $f = g$, i.e. iff $a_1 = 1/\overline{a_1}$ and $a_2 = \overline{a_2}/\overline{a_1}$. It is easy to check that

$$E := \{(a_1, a_2) \in \mathbb{C}^2 : a_1 = 1/\overline{a_1}, a_2 = \overline{a_2}/\overline{a_1}\} = \{(\zeta^2, x\zeta) : x \in \mathbb{R}, \zeta \in \mathbb{C}, |\zeta| = 1\}$$

and $(\zeta_1^2, x_1\zeta_1) = (\zeta_2^2, x_2\zeta_2)$ iff $(\zeta_1, x_1) = \pm(\zeta_2, x_2)$.

If $(a_1, a_2) \in E$, the non-degenerate admissible \mathbb{C} -form $\widehat{\Phi}$ corresponding to the isomorphism $\Phi : V \rightarrow V^\sharp$, $\Phi(e_{ks}) = f_{ks}$ is

$$\widehat{\Phi}(e_{ks}, e_{lr}) = (\Phi(e_{ks}))(e_{lr}) = f_{ks}(e_{lr}).$$

Using (6.6) and that $(e_{ks}^\sharp)(e_{lr}) = \delta_{kl}\delta_{sr}$, and explicit matrix for $\widehat{\Phi}$ in the basis $\{e_{ks}\}$ can be written down. As a curious aside we mention that the zero-set of the determinant of the symmetrized form $\widehat{\Phi} + \widehat{\Phi}^\sharp$ as a function of $z \in \mathbb{C} \setminus \{1\}$ via $a_2 = 1 - z$, $a_1 = (1 - z)/(1 - \bar{z})$ is the curve known as the *limaçon trisectrix*. It has certain special geometric properties and is parametrized in polar coordinates by $r = 1 + 2\cos\theta$. Thus, for points outside of this curve, $\widehat{\Phi} + \widehat{\Phi}^\sharp$ is the unique symmetric non-degenerate admissible form, by Remark 3.22.

Acknowledgements

The author would like to thank L. Turowska for many interesting discussions and helpful comments.

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